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Polynomial Parents of Some Binomial Congruences and Fermat Quotients

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Abstract: The main purpose of this article is to observe and use some rather simple polynomial identities to deduce some congruences involving binomial coefficients and Fermat quotients. Thus, the consequential identities or congruences may consider the polynomial identity as a parent. We point out that our purpose here is modest, in observing how some simple polynomial identities can lead to congruences for Fermat quotients etc. Indeed, some more general congruences are known in literature which connect Fermat quotients with Bernoulli numbers and harmonic sums, etc. In what follows, we do not spell out which of our identities are known in more general forms by other methods. This article is partly expository.

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1 Introduction

The starting point is almost two centuries back when Eisenstein proved a congruence satisfied by the Fermat quotient $\frac{2^{p-1}-1}{p}$ which also mysteriously appears in Wieferich's criterion in the first case of Fermat's last theorem. Since then Glaisher, Granville, Jothilingam, Emma Lehmer, Mestrovic, Ribenboim, Z.-H. Sun, Tauraso etc. have proved many other interesting congruences. We stress at the outset that all the polynomial identities in this article are elementary; it is the way of applying them that may be of interest. For these, the interested reader may consult [3], [2], [1], [4], [6], [7] and the references within them.

2 Congruences for Fermat quotients mod p

Eisenstein had proved for an odd prime that, modulo p, we have

$$\frac{2^{p-1}-1}{p} \equiv \sum_{j:odd,\ j<p-1} \frac{1}{j}.$$

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In what follows, for rational numbers $\frac{a}{b}, \frac{c}{d}$ and a prime number p not dividing the denominators b,d, one writes the congruence $\frac{a}{b} \equiv \frac{c}{d}$ modulo p to mean that $p|(ad-bc)$. It is possible to deduce Eisenstein's congruence, Glaisher's congruence, and some others by exploiting the polynomial identity below, which is self-evident.

Polynomial Identity I. For any integer $m \geq 1$, we have the polynomial identity

$$\frac{x^m - x}{m} = \sum_{r=1}^{m-1} \binom{m}{r} \frac{(x+1)^r(-1)^{m-r}}{m} + \frac{(x+1)^m - x + (-1)^m}{m}. \tag{1}$$

Lemma 2.1. *For any integer $n \geq 2$ and any odd prime p, we have the following congruences modulo p:*

$$\frac{n^p - n}{p} \equiv -\sum_{r=1}^{p-1} \frac{1^r + 2^r + \cdots + n^r}{r}. \tag{2}$$

$$\frac{n^p - n}{p} \equiv -\sum_{r=1}^{p-1} \frac{(-1)^r(1^r + 2^r + \cdots + (n-1)^r)}{r}. \tag{3}$$

In particular, modulo p, we have:

$$\frac{2^{p-1}-1}{p} \equiv -\frac{1}{2}\sum_{j=1}^{p-1} \frac{2^j}{j} \equiv -\frac{1}{2}\sum_{j=1}^{p-1} \frac{(-1)^j}{j} \equiv \frac{1}{2}\sum_{j=0}^{p-2} \frac{1}{\binom{p-2}{j}}. \tag{4}$$

$$\frac{2^{p-1}-1}{p} \equiv \sum_{j:\text{odd},\ j<p-1} \frac{1}{j} \equiv -\frac{1}{2}\sum_{j=1}^{(p-1)/2} \frac{1}{j}. \tag{5}$$

Proof. It follows from (1) for $m = p$ that:

$$\frac{a^p - a}{p} = \frac{(a+1-1)^p - a}{p} = \sum_{r=1}^{p-1} \binom{p}{r} \frac{(a+1)^r(-1)^{p-r}}{p} + \frac{(a+1)^p - (a+1)}{p}.$$

We claim that, modulo p,

$$\frac{a^p - a}{p} \equiv \sum_{r=1}^{p-1} \frac{(a+1)^r}{r} + \frac{(a+1)^p - (a+1)}{p}. \tag{6}$$

In the above, we used the observation that if p is an odd prime and $0 < r < p$, then the integer $\frac{1}{p}\binom{p}{r} \equiv \frac{(-1)^{r-1}}{r}$ mod p. This is so because

$$\frac{1}{p}\binom{p}{r} = \frac{(p-1)(p-2)\cdots(p-r+1)}{r!} \equiv \frac{(-1)^{r-1}(r-1)!}{r!} = \frac{(-1)^{r-1}}{r}.$$

Putting $a = 0$ gives the well-known congruence

$$\sum_{r=1}^{p-1} \frac{1}{r} \equiv 0 \ \ mod\ p.$$

Thus, the second congruence in (4) is an equivalent version of Eisenstein's congruence, which is the first one in (5).

To prove the third congruence in (4), note that

$$\frac{1}{\binom{p-2}{j}} = \frac{j!}{(p-2)(p-3)\cdots(p-1-j)} \equiv \frac{j!}{(-1)^j(j+1)!} = \frac{(-1)^j}{j+1}.$$

Summing over j from 0 to $p-2$ gives (with $j-1$ in place of j) the sum $\sum_{j=1}^{p-1} \frac{(-1)^{j-1}}{j}$ which is the second congruence of (4).

Putting $a = 1$ and $a = -2$ in the first congruence gives two of the congruences in the Lemma. Inductively, from (6), one gets then that

$$\frac{n^p - n}{p} \equiv -\sum_{r=1}^{p-1} \frac{2^r + \cdots + n^r}{r} \mod p.$$

When p is an odd prime, $\sum_{r=1}^{p-1} \frac{1}{r} \equiv 0 \mod p$ (indeed, it is even zero modulo p^2 when $p > 3$ by Wolstenholme's theorem). Thus, we have the more symmetric form asserted as (2). Finally, (3) is gotten similarly to (2) inductively from (6) by putting $a = -2, -3, -4$ etc.

Clearly, using the fact that $\sum_{j \leq p-1} \frac{1}{j} \equiv 0 \mod p$, one has the two congruences in (5). This completes the proof of the lemma. □

3 Congruences modulo p^2

We prove a couple of polynomial identities, which are then used to obtain different congruences for Fermat quotients modulo p^2. Many authors (for example, see [6], [4], [3]) have obtained similar and, sometimes, more general congruences.

Lemma 3.1. Polynomial Identity II. *For any odd positive integer n,*

$$\sum_{r=1}^{n} \frac{(-1)^{r-1} x^r}{r} = \sum_{r=1}^{n} \binom{n}{r} \frac{(-1)^{r-1}(x+1)^r}{r} + \sum_{r=1}^{n} \frac{(-1)^r}{r} \binom{n}{r}.$$

Proof. Consider the evident identity

$$x^n = \sum_{r=1}^{n} \binom{n}{r} (x+1)^r (-1)^{r-1} - 1.$$

As n is odd, we have

$$1 - x + x^2 - \cdots + x^{n-1} = \frac{x^n + 1}{x+1} = \sum_{r=1}^{n} \binom{n}{r} (-1)^{r-1} (x+1)^{r-1}.$$

Integration gives

$$x - \frac{x^2}{2} + \cdots + \frac{x^n}{n} = \sum_{r=1}^{n} \binom{n}{r} (-1)^{r-1} \frac{(x+1)^r}{r} + c,$$

for some constant c. Putting $x = 0$, we obtain $c = \sum_{r=1}^{n} \frac{(-1)^r}{r} \binom{n}{r}$. Thus, we have the asserted polynomial identity:

$$\sum_{r=1}^{n} \frac{(-1)^{r-1} x^r}{r} = \sum_{r=1}^{n} \binom{n}{r} \frac{(-1)^{r-1}(x+1)^r}{r} + \sum_{r=1}^{n} \frac{(-1)^r}{r} \binom{n}{r}.$$

□

Note, in passing, that by comparing coefficients of x^2 and x^3, we get

$$n = \sum_{r=1}^{n-1} (-1)^r \binom{n}{r} (r-1);$$

$$(n-1)(n-2) = 2 + \sum_{r=1}^{n-1} (-1)^r \binom{n}{r} (r-1)(r-2).$$

We now use the above polynomial identity to deduce a congruence for the Fermat quotient modulo p^2.

Proposition 3.2. *For any prime $p > 3$,*

$$\frac{2^{p-1}-1}{p} \equiv \sum_{r=1}^{p-1} \frac{-2^{r-1}}{r} \mod p^2.$$

Proof. In the polynomial identity

$$\sum_{r=1}^{n} \frac{(-1)^{r-1}x^r}{r} = \sum_{r=1}^{n} \binom{n}{r} \frac{(-1)^{r-1}(x+1)^r}{r} + \sum_{r=1}^{n} \frac{(-1)^r}{r} \binom{n}{r},$$

take $x = -2$. We obtain

$$-\sum_{r=1}^{p} \frac{2^r}{r} = -\sum_{r=1}^{P-1} \binom{p}{r} \frac{1}{r} - \frac{2}{p} + \sum_{r=1}^{p-1} \binom{p}{r} \frac{(-1)^r}{r}.$$

Rewriting this, we have

$$\frac{2^p-2}{p} + \sum_{r=1}^{p-1} \frac{2^r}{r} = 2 \sum_{r:\text{odd},\ r<p} \binom{p}{r} \frac{1}{r} \cdots (\diamondsuit)$$

Therefore, the proposition will follow if we show that for $p \geq 5$,

$$\sum_{r:\text{odd},\ r<p} \binom{p}{r} \frac{1}{pr} \equiv 0 \mod p.$$

Now for $1 \leq r \leq p-2$ with r odd, we have, modulo p,

$$\frac{1}{pr} \binom{p}{r} = \frac{(p-1)(p-2)\cdots(p-r+1)}{r^2} \equiv \frac{1}{r^2}$$

since r is odd. Therefore,

$$\sum_{r:\text{odd},\ r<p} \binom{p}{r} \frac{1}{pr} \equiv \sum_{r:\text{odd},\ r<p} \frac{1}{r^2} \equiv \frac{1}{2} \sum_{r=1}^{p-1} \frac{1}{r^2}$$

since $(p-r)^2 \equiv r^2 \mod p$ and $p-r$ runs through the even integers $< p$ when r runs through the odd integers $< p$. But,

$$\sum_{r=1}^{p-1} \frac{1}{r^2} \equiv \sum_{d=1}^{p-1} d^2 = \frac{(p-1)p(2p-1)}{6} \equiv 0$$

if $p \geq 5$. Therefore, the proposition is proved. □

A striking consequence of the polynomial identity II is an identity involving the harmonic numbers; this is:

Corollary 3.3. *(of Polynomial Identity)*

$$H_n := \sum_{k=1}^{n} \frac{1}{k} = \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k+1}}{k}.$$

Proof. The polynomial identity can be rephrased as

$$\sum_{k=1}^{n} \frac{(1-x)^k}{k} = \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k(x^k-1)}{k},$$

and, can be integrated to yield

$$\sum_{k=1}^{n} \frac{(1-x)^{k+1}}{k(k+1)} = \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k+1}}{k} \left(\frac{x^{k+1}}{k+1} - x\right) + C,$$

where we get $C = \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k+1}}{k+1}$ by putting $x = 1$. Equating the coefficients of x on both sides of the polynomial identity gives us the identity:

$$H_n := \sum_{k=1}^{n} \frac{1}{k} = \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k+1}}{k}.$$

Here are some more consequences of the same polynomial identity used in the proposition.

Lemma 3.4. *For all positive integers n, the following identities hold good:*

$$\sum_{r=1}^{n} \frac{2^r - 1}{r} = \sum_{r=1}^{n} \binom{n}{r} \frac{1}{r};$$

$$\sum_{r=1}^{n} \frac{2^r - r - 1}{r(r+1)} = \sum_{r=1}^{n} \binom{n}{2r} \frac{1}{2r(2r+1)}.$$

Also, we have the polynomial identity:

$$y \sum_{r=1}^{n} \frac{y^r - r - 1}{r(r+1)} = \sum_{r=1}^{n} \binom{n}{r} \frac{(y-1)^{r+1} - (-1)^{r+1}}{r(r+1)}.$$

Proof. Again, consider the polynomial identity

$$\sum_{r=1}^{n} \frac{(-1)^{r-1} x^r}{r} = \sum_{r=1}^{n} \binom{n}{r} \frac{(-1)^{r-1}(x+1)^r}{r} + \sum_{r=1}^{n} \frac{(-1)^r}{r} \binom{n}{r}.$$

Using

$$\sum_{k=1}^{n} \frac{1}{k} = \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k+1}}{k},$$

the polynomial identity can be rewritten (with y in place of $-x$) as:

$$\sum_{r=1}^{n} \frac{y^r - 1}{r} = \sum_{r=1}^{n} \binom{n}{r} \frac{(y-1)^r}{r}.$$

Putting $y = 2$ in this gives us the first identity of the lemma. Finally, integrating it and determining the constant of integration, one obtains the polynomial identity:

$$y \sum_{r=1}^{n} \frac{y^r - r - 1}{r(r+1)} = \sum_{r=1}^{n} \binom{n}{r} \frac{(y-1)^{r+1} - (-1)^{r+1}}{r(r+1)}.$$

Evaluating the polynomial identity at $y = 2$ gives us the second identity in the lemma. □

Corollary 3.5. *(Consequence of Z.-H. Sun's congruence).*
For any odd prime p, we have

$$\sum_{i=0}^{p-1} \frac{1}{\binom{p-1}{i}} \equiv \frac{1}{2^{p-1}} \left(1 - \frac{7p^3 B_{p-3}}{24}\right) \ mod \ p^4$$

where B_{p-3} is the $(p-3)$-th Bernoulli number.

Proof. Here, we use a beautiful, nontrivial congruence due to Z.-H. Sun, who proved in 2008, that

$$\frac{2^p - 2}{p} + \sum_{r=1}^{p-1} \frac{2^r}{r} = \frac{-7}{12} p^2 B_{p-3} \mod p^3.$$

In [9], we had shown for any positive integer n that

$$\sum_{i=0}^{n} \frac{1}{\binom{n}{i}} = \frac{n+1}{2^n} \sum_{j\ odd} \binom{n+1}{j} \frac{1}{j}.$$

Applying this to $n = p-1$ for an odd prime p, we have

$$\sum_{i=0}^{p-1} \frac{1}{\binom{p-1}{i}} = \frac{1}{2^{p-1}} + \frac{p}{2^{p-1}} \sum_{j\ odd, j<p} \binom{p}{j} \frac{1}{j}.$$

Recall that earlier from a polynomial identity, we had observed for any odd prime p that

$$\frac{2^p-2}{p} + \sum_{r=1}^{p-1} \frac{2^r}{r} = 2 \sum_{r:\text{odd},\ r<p} \binom{p}{r} \frac{1}{r} \cdots (\diamondsuit)$$

Combining this with Sun's congruence, and using $(\diamondsuit)$, we obtain that

$$\sum_{rodd, r<p} \binom{p}{r} \frac{1}{pr} \equiv \frac{-7}{24} pB_{p-3} \ \ mod \ \ p^2.$$

Therefore, we have the assertion of the corollary. □

Corollary 3.6. *For odd primes $p \geq 5$,*

$$\sum_{j\ odd, j<p} \binom{p}{j} \frac{1}{j} \equiv \frac{7}{8} \sum_{r=1}^{p-1} \frac{1}{r} \ \ mod \ \ p^3.$$

Proof. Let $p > 3$ be a prime. The sum $\sum_{r=1}^{p-1} \frac{1}{r}$ is known to be $\frac{-p^2}{3} B_{p-3}$ mod p^4, the identity $(\diamondsuit)$ and Sun's identity above imply the assertion. □

Remark. It would be interesting to find an elementary proof of the Corollary without using Sun's rather nontrivial result.

We have already proved a congruence for $\frac{2^p-2}{p}$ modulo p^2. In 1985, Jothilingam [4] proved a congruence that involves an ordered choice of quadratic residues. Below, we prove a different, simpler congruence. The authors in [3] have also proved such a congruence, using the polynomial identity appearing below.

Lemma 3.7. *(Polynomial Identity III.)*

$$\sum_{r=0}^{n-1} \frac{(1-x)^r}{r+1} = \sum_{r=0}^{n-1} \binom{n}{r+1} \frac{(-1)^r}{r+1} \frac{x^{r+1}-1}{x-1}.$$

Proof. Start with the elementary polynomial identity

$$-\sum_{k=1}^{n} (1-x)^{k-1} = \sum_{k=1}^{n} \binom{n}{k} (-1)^k x^{k-1}.$$

Integrating this, we have

$$\sum_{k=1}^{n} \frac{(1-x)^k}{k} = \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k (x^k-1)}{k}.$$

The above identity has been written after finding the constant of integration by putting $x = 1$. Rewriting the above identity by taking $k = r+1$, we have the asserted polynomial identity:

$$\sum_{r=0}^{n-1} \frac{(1-x)^r}{r+1} = \sum_{r=0}^{n-1} \binom{n}{r+1} \frac{(-1)^r}{r+1} \frac{x^{r+1}-1}{x-1}. \tag{7}$$

As an application of the polynomial identity III, we get:

Theorem 3.8. *For any $n \geq 1$, we have*

$$\sum_{r=1}^{n} \frac{2^r}{r^2} = 2 \sum_{r=1}^{n} \binom{n}{r} \frac{(-1)^r}{r} \sum_{k\ odd, k \leq r} \frac{1}{k}.$$

In particular, for any odd prime p, we have

$$\frac{2^{p-1}-1}{p} - \sum_{r:\text{odd},\ r<p} \frac{1}{r} \equiv -p \sum_{r=1}^{p-1} \frac{2^{r-1}}{r^2} \mod p^2.$$

Proof. For any $n \geq 1$, integrating (7) we have

$$\sum_{r=0}^{n-1} \frac{(1-x)^{r+1}}{(r+1)^2} = \sum_{r=0}^{n-1} \binom{n}{r+1} \frac{(-1)^{r+1}}{r+1} \left(x + \frac{x^2}{2} + \cdots + \frac{x^{r+1}}{r+1} \right) + C,$$

where the constant C is obtained by putting $x = 1$ when the left hand side is 0. We obtain

$$C = \sum_{r=0}^{n-1} \binom{n}{r+1} \frac{(-1)^r}{r+1} \left(1 + \frac{1}{2} + \cdots + \frac{1}{r+1} \right).$$

Let us consider $x = -1$ and $n = p$, for an odd prime p. We have

$$\sum_{r=0}^{p-1} \frac{2^{r+1}}{(r+1)^2} = \sum_{r=0}^{p-1} \binom{p}{r+1} \frac{(-1)^{r+1}}{r+1} \left(x + \frac{x^2}{2} + \cdots + \frac{x^{r+1}}{r+1} \right)$$
$$+ \sum_{r=0}^{p-1} \binom{p}{r+1} \frac{(-1)^r}{r+1} \left(1 + \frac{1}{2} + \cdots + \frac{1}{r+1} \right).$$

For $r < p-1$, since $\binom{p}{r+1} \equiv 0 \mod p$, we have the corresponding sub-sum on the right-hand side to be

$$\sum_{r=0}^{p-2} \binom{p}{r+1} \frac{(-1)^{r+1}}{r+1} \left(x + \frac{x^2}{2} + \cdots + \frac{x^{r+1}}{r+1} \right) \equiv 0 \ mod \ p.$$

The single term corresponding to $r = p-1$ is (as $\binom{p}{r+1} = 1$)

$$\frac{(-1)^p}{p} \sum_{d \leq p} \frac{(-1)^d}{d} + \frac{(-1)^{p-1}}{p} \sum_{d \leq p} \frac{1}{d} = \frac{-1}{p} \sum_{d \leq p} \frac{(-1)^d - 1}{d} = \frac{2}{p} \sum_{d\ odd; d \leq p} \frac{1}{d}.$$

Therefore,

$$\sum_{r=0}^{p-1} \frac{2^{r+1}}{(r+1)^2} \equiv \frac{2}{p} \sum_{d\ odd; d \leq p} \frac{1}{d} \mod p.$$

Clearly, this is the congruence

$$\frac{2-2^p}{p^2} \equiv \sum_{r=1}^{p-1} \frac{2^r}{r^2} - \frac{2}{p} \sum_{r:\text{odd},\ r<p} \frac{1}{r} \mod p,$$

which gives, on multiplying by p, the asserted congruence modulo p^2 in the theorem. □

Corollary 3.9. *(of Granville's Congruence) For any prime $p \geq 5$,*

$$\sum_{r odd, r<p} \frac{1}{r} \equiv \frac{2^{p-1}-1}{p} - \frac{p}{2} \left(\frac{2^{p-1}-1}{p} \right)^2 \ mod \ p^2.$$

Proof. The proof follows from the theorem above and Granville's congruence which asserts

$$\left(\frac{2^{p-1}-1}{p} \right)^2 \equiv -\sum_{r=1}^{p-1} \frac{2^r}{r^2} \ mod \ p$$

for any prime $p > 3$. □

4 More identities for Fibonacci, Lucas, and Bernoulli numbers

In the course of the proof of a lemma in the previous section, we had rewritten the polynomial identity II as:

$$\sum_{r=1}^{n}\frac{y^r-1}{r}=\sum_{r=1}^{n}\binom{n}{r}\frac{(y-1)^r}{r}.$$

Here are some interesting consequences that we point out:

Lemma 4.1. *For any positive integer n, we have*

$$\sum_{r=1}^{n}\frac{\beta^{n-r}}{r}\left(F_r-\binom{n}{r}5^{(r-1)/2}\right)=0,$$

where F_r is the r-th Fibonacci number, and $\beta=\frac{1-\sqrt{5}}{2}$. Further, if B_k denotes the k-th Bernoulli number, we have for any positive integers n,N:

$$\sum_{r=1}^{n}\frac{1}{r(r+1)}\sum_{\ell=0}^{r}\binom{r+1}{\ell}(N+1)^{r+1-\ell}B_\ell$$

$$=N\sum_{r=1}^{n}\frac{1}{r}+\sum_{r=1}^{n}\frac{\binom{n}{r}}{r(r+1)}\sum_{\ell=0}^{r}\binom{r+1}{\ell}N^{r+1-\ell}B_\ell.$$

Proof. The identity involving Fibonacci numbers is proved by taking $y=\alpha/\beta$, where $\alpha=\frac{1+\sqrt{5}}{2}$ and using the well-known identity $F_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}$.

To obtain the second identity, take repeatedly $y=1,2,\cdots,N$ in the polynomial identity and use the well-known relation

$$\frac{1}{r+1}\sum_{\ell=0}^{r}\binom{r+1}{\ell}(N+1)^{r+1-\ell}B_\ell=1^r+2^r+\cdots+N^r.$$

□

Earlier, we (as well as many others) have used the polynomial identity:

Polynomial Identity IV.

$$\sum_{i\geq 0}(-1)^i\binom{n-i}{i}(xy)^i(x+y)^{n-2i}=\frac{x^{n+1}-y^{n+1}}{x-y}.$$

Among other things, one can deduce the classical identities for Fibonacci numbers:

$$F_{n+1}=\sum_{i\geq 0}\binom{n-i}{i}=\prod_{r=1}^{[(n-1)/2]}(3+2\cos 2\pi r/n).$$

This allows us to immediately deduce $F_m|F_n$ if $m|n$ - see [8]. We mention in passing that, in [5], certain polynomial identities in several variables are obtained which leads to expressions for powers of a matrix and for multi-Fibonacci numbers. The following expression for the Lucas numbers is similar in spirit to the first expression for F_n, is a consequence of the above polynomial identity:

Lemma 4.2.

$$L_{2n}=2n\sum_{i\geq 0}(-1)^i\binom{2n-i}{i}\frac{5^{n-i}}{2n-i};$$

$$L_{2n+1}=(2n+1)\sum_{i\geq 0}\binom{2n+1-i}{i}\frac{1}{2n+1-i}.$$

These expressions for the L_n's can also presumably be deduced using the corresponding expression for the F_n's above. One can integrate the above polynomial identity (written in one variable x as):

$$\sum_{i\geq 0}(-1)^i\binom{n-i}{i}(x^2-x)^i = \frac{(1-x)^{n+1}-x^{n+1}}{1-2x}$$

to obtain:

Lemma 4.3. *(Polynomial Identity V.) We have the polynomial identity*

$$\sum_{i\geq 1} i\binom{n-i}{i}(x^2-x)^{i-1} = \frac{2(1-x)^{n+1}-2x^{n+1}-(n+1)(1-2x)[(1-x)^n+x^n]}{(2x-1)^3}.$$

In particular, we have the identity:

$$\sum_{i\geq 1} 2^{i-1} i\binom{n-i}{i} = \frac{2(-1)^{n+1}-2^{n+2}+3(n+1)[2^n+(-1)^n]}{27}.$$

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