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On the Positive Roots of Positive Real Numbers

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Abstract: In this article, we find an interesting property of the $(n+k)^{\text{th}}$ root of n, a natural number. Then we generalize it to the real numbers and study its possible applications.

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1 Introduction

The k^{th} root of a positive real number is very often irrational. This phenomenon has attracted many experts and amateurs to produce beautiful mathematical results in this direction. History shows that the problem of finding such irrational values, correct to a certain digit after the decimal point, has been analyzed in almost every continent in the Middle Ages [1]. For example, work of Aryabhatta and Archimedes on finding the square roots could be found respectively in [6] and [4]. The method we use nowadays to calculate the roots of integers is the "Newton-Raphson" method. A brief overview of the historical development of this method can be found in [5]. Although this method is used as a strong tool in modern sciences it has some limitations and many improvisations have been made to it. Today, finding an algorithm better than this and increasing its efficiency are two challenges in numerical analysis.

While finding the k^{th} root using "Newton-Raphson" method the graph of the function $f(n) = n^{1/k}$, where k is fixed, is considered. That is, the power is fixed and the base is a variable. A question generally arises what the graph of the function $f(n) = n^{1/(n+k)}$ would look like? That is, the base is a variable as well as the power. We find here that the graph of this function increases up to a certain point and then decreases. Interestingly enough, this property is nothing but a generalization of the following old Indian Olympiad (RMO) problem:

$$\text{If } n \geq 3, \text{ then } n^{(n+1)} > (n+1)^n. \tag{1.1}$$

In the second section, we derive some results through which we make a conjecture on the behavior of the function, $f(n) = n^{1/n+k}$. In the third section, we analyze this function and prove our conjecture. The discussion in this paper gives hopes of finding an algorithm to determine the m^{th} root of an integer n, where $m \in \mathbb{N}$ is the set of natural numbers.

2 Preliminaries

In this section, we study some inequalities that give hints about the m^{th} root of n. Unless otherwise mentioned, here we assume $n, m, a, b, k \in \mathbb{N}$.

Theorem 2.1. *For $n \geq m \geq 2$, $n^m > (n+1)^{m-1}$.*

Proof. The right-hand side can be expanded using the binomial theorem for $m \geq 2$,

$$(n+1)^{m-1} = n^{m-1} + \binom{m-1}{1} n^{m-2} + \binom{m-1}{2} n^{m-3} + \cdots + \binom{m-1}{m-2} n + 1.$$

Now we can remove the term n^{m-1} from each side of the inequality. Then the inequality that we have to prove reduces to,

$$(n-1)n^{m-1} > \binom{m-1}{1} n^{m-2} + \binom{m-1}{2} n^{m-3} + \cdots + \binom{m-1}{m-2} n + 1. \tag{2.1}$$

As $n \geq m$,

$$n > (m-1), (m-2), \ldots, (m-1-(k-1)).$$

Hence $n^k > \binom{m-1}{k}$. This implies $n^{m-1} > \binom{m-1}{k} n^{m-1-k}$ for all $1 \leq k \leq m-1$. This means that each term on the right-hand side of (2.1) is less than n^{m-1}. Since $n \geq m$, there are more or equal numbers of n^{m-1}'s on the left than the number of terms on the right. Hence we can say that for all $n \geq m$, the inequality (2.1) holds. That is $n^m > (n+1)^{m-1}$ for all $n \geq m \geq 2$. □

Corollary 2.2. *If $n \geq m \geq 2$ then, $n^{1/(m-1)} > (n+1)^{1/m}$.*

Proof. Follows directly from Theorem 2.1. □

Corollary 2.3. *If $a > b > k \geq 1$, then $a^{1/(a-k)} < b^{1/(b-k)}$.*

Proof. The proof can be divided into two cases: $k = 1$ and $k > 1$.
Case I: $k = 1$. Putting $n = m = b$ in Corollary 2.2 we get, $b^{1/(b-1)} > (b+1)^{1/((b+1)-1)}$ for any $b \geq 2$. This generates the following chain of inequalities

$$b^{1/(b-1)} > (b+1)^{1/((b+1)-1)} > (b+2)^{1/((b+2)-1)} \cdots > (b+k')^{1/((b+k')-1)} \cdots$$

Now, $b > k = 1$ forces b to be at least 2. Also, $a = b + k'$ for some k'. Hence the result.
Case II: $k > 1$. Let $a = b + k'$, where $k' \in \mathbb{N}$. As $b > k \geq 1$ we have the following inequalities,

$$b \geq (b-k+1), (b+1) \geq (b-k+2), \ldots, (b+k') \geq (b+k'-k+1).$$

Now applying Corollary 2.2 to each of these $k'+1$ inequalities produces another $k'+1$ inequalities. These are

$$b^{1/(b-k)} > (b+1)^{1/(b-k+1)}, (b+1)^{1/(b-k+1)} > (b+2)^{1/(b-k+2)}, \ldots$$

$$(b+k'-1)^{1/(b+k'-k-1)} > (b+k')^{1/(b+k'-k)}, (b+k')^{1/(b+k'-k)} > (b+k'+1)^{1/(b+k'-k+1)}$$

Now putting these together produces $b^{1/(b-k)} > (b+k')^{1/(b+k'-k)}$ and hence we have the result. □

Collecting the Problem 1.1, mentioned in the previous section, as a theorem here, (for proof see [2]).

Theorem 2.4. *If $n \geq 3$, then $n^{(n+1)} > (n+1)^n$.*

Similar to Corollaries 2.2 and 2.3, we have the following corollaries.

Corollary 2.5. *If $n \geq 3$ then, $n^{1/n} > (n+1)^{1/(n+1)}$.*

Corollary 2.6. *If $a > b \geq 3$ then $a^{1/a} < b^{1/b}$.*

Lemma 2.7. *For all $n \geq 4$,*

$$n^{n+1} > \binom{n}{n-2} n^3 + \binom{n}{n-1} n^2 + n.$$

Proof. On dividing both sides by n, the inequality becomes $n^n > n^2(\frac{n(n-1)+2}{2}) + 1$. Now let $\frac{n(n-1)+2}{2} \geq n^2$. This implies $n(n-1)+2 \geq 2n^2$ or $2-n \geq n^2$. Which is impossible for all $n \geq 4$. So $\frac{n(n-1)+2}{2} < n^2$. That is $n^2(\frac{n(n-1)+2}{2}) < n^4$. Again let $n^2(\frac{n(n-1)+2}{2}) + 1 = n^4$. This will produce $2n^2 - n^3 + 1 = n^4$. This is again absurd; because for all $n \geq 4, 2n^2 < n^3$. Hence we can say that $n^4 > n^2(\frac{n(n-1)+2}{2}) + 1$. So $n^n \geq n^4 > n^2(\frac{n(n-1)+2}{2}) + 1$. □

Theorem 2.8. *For all $n \geq 4$, $n^{n+2} > (n+1)^{n+1}$.*

Proof. The left hand side is equal to $n.n^{n+1} = n^{n+1} + n^{n+1} + \cdots + n^{n+1}$. The right hand side is $(n+1)^{n+1} = (n+1)(n+1)^n = n(n+1)^n + (n+1)^n$. Using the binomial theorem this can be expanded to get the following :

$$n\left(n^n + \binom{n}{1} n^{n-1} + \binom{n}{2} n^{n-2} + \cdots + \binom{n}{n-1} n + 1\right) + (n+1)^n.$$

Now the inequality becomes,

$$\underbrace{n^{n+1} + \cdots + n^{n+1}}_{n\,times} > n^{n+1} + \binom{n}{1} n^n + \binom{n}{2} n^{n-1} + \cdots + \binom{n}{n-1} n^2 + n + (n+1)^n.$$

Let's rearrange the terms on the right-hand side into three distinct parts viz I, II, and III.

$$\underbrace{\binom{n}{2} n^{n-1} + \cdots + \binom{n}{n-3} n^4 +}_{\text{I}} \underbrace{\binom{n}{n-2} n^3 + \binom{n}{n-1} n^2 + n}_{\text{II}} \qquad \underbrace{+ n^{n+1} + \binom{n}{1} n^n + (n+1)^n}_{\text{III}}$$

For I, notice that $n^k > \binom{n}{k}$, for all $2 \leq k \leq n$. So $n^{n+1} = n^k n^{n+1-k} > \binom{n}{k} n^{n+1-k}$. Hence,

$$(n-4)(n^{n+1}) > \binom{n}{2} n^{n-1} + \cdots + \binom{n}{n-3} n^4$$

For II, using Lemma 2.7 ,

$$n^{n+1} > \binom{n}{n-2} n^3 + \binom{n}{n-1} n^2 + n$$

For III, by Theorem 2.4,

$$3n^{n+1} > n^{n+1} + \binom{n}{1} n^n + (n+1)^n$$

Combining the last three inequalities produces the desired result. □

The following are now natural corollaries.

Corollary 2.9. *If $n \geq 4$ then, $n^{1/(n+1)} > (n+1)^{1/(n+2)}$.*

Corollary 2.10. *If $a > b \geq 4$ then $a^{1/(a+1)} < b^{1/(b+1)}$.*

Lemma 2.11. *For all $n \geq 5$,*

$$n^{n+2} > \binom{n+1}{n-2} n^4 + \binom{n+1}{n-1} n^3 + \binom{n+1}{n} n^2 + n + n^{n+1}.$$

Proof. The right-hand side is equal to,

$$\frac{(n+1)n(n-1)}{6}n^4 + \frac{n(n+1)}{2}n^3 + (n+1)n^2 + n + n^{n+1}.$$

This is again equal to

$$((n+1)n(n-1)n^4 + 3n(n+1)n^3 + 6(n+1)n^2 + 6n + 6n^{n+1})/6.$$

On multiplying both sides by 6 the inequality becomes,

$$6n^{n+2} > (n+1)n(n-1)n^4 + 3n(n+1)n^3 + 6(n+1)n^2 + 6n + 6n^{n+1} \text{ or}$$

$$6n^{n+2} > n^7 + 2n^5 + 3n^4 + 6n^3 + 6n^2 + 6n + 6n^{n+1}.$$

On removing n, it reduces to $6n^{n+1} > n^6 + 2n^4 + 3n^3 + 6n^2 + 6n + 6 + 6n^n$.

Assume that for some $n \geq 5$, $6 + 6n^n \geq 2n^{n+1}$. This will imply $3 + 3n^n \geq n^{n+1}$ or $3 \geq n^n(n-3)$. Now the last inequality is absurd for all $n \geq 4$. Hence $2n^{n+1} > 6 + 6n^n$. Similarly assume that for some $n \geq 5$, $6n^2 + 6n \geq n^{n+1}$. Again this will imply $6n + 6 \geq n^n$ or $6 \geq n(n^{n-1} - 6)$. This is also impossible, because for all $n \geq 5$, $n(n^{n-1} - 6) \geq 5.(5^4 - 6) > 6$. Hence we can say that for all $n \geq 5$, $n^{n+1} > 6n^2 + 6n$. Moreover for all $n \geq 4$ we have, $n^{n+1} \geq n^6$ and $n^{n+1} > 2n^4, 3n^3$.

Putting these pieces together, we get for all $n \geq 5$,

$$6n^{n+1} > n^6 + 2n^4 + 3n^3 + 6n^2 + 6n + 6 + 6n^n.$$

That is

$$n^{n+2} > \binom{n+1}{n-2}n^4 + \binom{n+1}{n-1}n^3 + \binom{n+1}{n}n^2 + n + n^{n+1}.$$

This completes the proof. □

Theorem 2.12. *For all $n \geq 5$, $n^{n+3} > (n+1)^{n+2}$.*

Proof. Like the proof of Theorem 2.8 we get the following equivalent inequality.

$$n.n^{n+2} > n^{n+2} + \binom{n+1}{1}n^{n+1} + \binom{n+1}{2}n^n + \cdots + \binom{n+1}{n}n^2 + n + (n+1)^{n+1}$$

Now for $2 \leq k \leq n$, $k! = 2m$ for some $m \in \mathbb{N}$. But $n > \frac{(n+1)}{2}$, for all $n \geq 3$. Moreover $n^{k-1} > n.(n-1)\cdots(n+1-(k-1))$. These implies

$$n^k > \frac{n+1}{2}.n\cdots(n+1-(k-1)) \geq \binom{n+1}{k}$$

for all $2 \leq k \leq n$. Using this we can have $n^{n+2} > \binom{n+1}{k}.n^{n+2-k}$. Applying this result to all $2 \leq k \leq (n-3)$ gives, $(n-4)n^{n+2} > \binom{n+1}{2}n^n + \cdots + \binom{n+1}{n-3}$.

Now by Theorem 2.8 we have $3n^{n+2} > n^{n+2} + \binom{n+1}{1}n^{n+1} + (n+1)^{n+1}$. Moreover Lemma 2.11 gives, $n^{n+2} > \binom{n+1}{n-2}n^4 + \binom{n+1}{n-1}n^3 + \binom{n+1}{n}n^2 + n$. Now adding the last three inequalities gives, for all $n \geq 5$,

$$n.n^{n+2} > n^{n+2} + \binom{n+1}{1}n^{n+1} + \binom{n+1}{2}n^n + \cdots + \binom{n+1}{n-3} + \binom{n+1}{n-2}n^4$$
$$+ \binom{n+1}{n-1}n^3 + \binom{n+1}{n}n^2 + n + (n+1)^{n+1}.$$

That is $n^{n+3} > (n+1)^{n+2}$.

□

Again we arrive at similar corollaries.

Corollary 2.13. *If $n \geq 4$ then $n^{1/(n+2)} > (n+1)^{1/(n+3)}$.*

Corollary 2.14. *If $a > b \geq 4$ then $a^{1/(a+2)} < b^{1/(b+2)}$.*

Major inequalities in this section can now be collected to see a general pattern.

- If $a > b \geq 2$ and $k \geq 1$ then $a^{1/(a-k)} < b^{1/(b-k)}$ [Corollary 2.3]
- If $a > b \geq 3$ and $k = 0$ then $a^{1/(a-k)} < b^{1/(b-k)}$ [Corollary 2.6]
- If $a > b \geq 4$ and $k = 1$ then $a^{1/(a+k)} < b^{1/(b+k)}$ [Corollary 2.10]
- If $a > b \geq 5$ and $k = 2$ then $a^{1/(a+k)} < b^{1/(b+2)}$ [Corollary 2.13]

It is evident that for a negative k, if $f_k : (-k, \infty) \to \mathbb{R}$ be a function defined by $f_k(n) = n^{1/(n+k)}$, where $(-k, \infty)$ denotes all the natural numbers in the interval, then f_k is a strictly decreasing function in $(-k, \infty)$. For $k \in \{0, 1, 2\}$ the function $f_k(n) = n^{1/(n+k)}$ is a decreasing function after a point m, on its domain (k, ∞). At this point, a conjecture arrives naturally. That is if we increase the value of k to any natural number greater than 2, then also $f_k(n) = n^{1/(n+k)}$ is a decreasing function for all $n \geq m$, where m is a unique integer in the domain of f_k. We shall prove this in the next section.

3 Properties of f_k

We state below some basic theorems before proceeding further. Then we define f_k not only for integers but also for positive reals.

Theorem 3.1. *Let f be a real function continuous on $[a, b]$ and differentiable on the open interval (a, b), where $a, b \in \mathbb{R}$. Then*

1. *f is strictly increasing in $[a, b]$ if $f'(x) > 0$ for each $x \in (a, b)$,*
2. *f is strictly decreasing in $[a, b]$ if $f'(x) < 0$ for each $x \in (a, b)$,*
3. *f is a constant function in $[a, b]$ if $f'(x) = 0$ for each $x \in (a, b)$.*

Proof. See [3]. □

Theorem 3.2. *If a function f is differentiable at a point, then it is also continuous at that point.*

Proof. See [3]. □

Definition 3.3. For any $k \in \mathbb{Z}$, q is defined to be the smallest positive integer such that $q + k \geq 0$. That is $q = 1$ for all $k \geq 0$ and $q = -k$ for all negative k.

Definition 3.4. Let $f_k : (q, \infty) \to \mathbb{R}$ be a function defined as $f_k(x) = x^{1/(x+k)}$ where $x \in \mathbb{R}$ and $k \in \mathbb{Z}$.

We have the following basic results about f_k.

Lemma 3.5. *The function f_k is differentiable on the interval (q, ∞).*

Proof. It is known that $e^{\ln(y)} = y$ for all $y \in \mathbb{R}+$. Now by the definition, $f_k(x) = x^{1/(x+k)} > 0$ where $x \in (q, \infty)$. Hence $f_k(x) = x^{1/(x+k)} = e^{\ln(x^{1/(x+k)})} = e^{\frac{\ln(x)}{x+k}}$.This implies that f_k is an exponential function. But all exponential functions are differentiable. Hence f_k is also differentiable. □

Lemma 3.6. *The function f_k is continuous on the real interval (q,∞).*

Proof. Lemma 3.5 says f_k is differentiable at each point of the interval (q,∞). Then by Theorem 3.2, f_k is also continuous at each point of the interval (q,∞). That is f_k is continuous on (q,∞). □

Before proceeding further we introduce another function g_k. Properties of this function will turn out to be very useful in understanding the properties of f_k.

Definition 3.7. Let $g_k : [q,\infty) \to \mathbb{R}$ be the function defined as $g_k(x) = \frac{x+k}{x} - \ln(x)$, where k is an integer.

Lemma 3.8. *The function g_k is differentiable on the interval $[q,\infty)$.*

Proof. Can be verified easily using basic properties of limits (see [3]). □

Lemma 3.9. *The function g_k is continuous on the interval $[q,\infty)$.*

Proof. The proof follows from Lemma 3.8 and Theorem 3.2. □

Lemma 3.10. *The function g_k is strictly decreasing in the interval $[q,\infty)$.*

Proof. Let $q < \delta < \infty$ be an arbitrary real number. Now by Lemma 3.8 and Lemma 3.9, g_k is differentiable on $[q,\delta]$ and g_k is continuous on (q,δ). we note that $\frac{d}{dx}(g_k(x)) = -\frac{x+k}{x^2}$, which is always negative. Hence by Theorem 3.1, g_k is strictly decreasing in $[q,\delta]$. This holds for all $q < \delta < \infty$. This completes the proof. □

Lemma 3.11.

1. *If $k \geq 0$, then there exists a real number $c \in (q,\infty)$ such that, $g_k(c) = 0$; $g_k(x) > 0$ for all $x < c$ and $g_k(x) < 0$ for all $x > c$.*
2. *If $k < 0$, then $g_k(x) < 0$ for all $x \in (q,\infty)$.*

Proof.

1. For $k \geq 0$, $q = 1$. Hence $g_k(q) = \frac{1+k}{1} - \ln(1) = 1+k$, which is positive. Now assume that the values of g_k are always positive. That is $g_k(x) > 0$ for all $x \in (1,\infty)$. Now let $y = e^{(k+2)}$. Hence $\ln(y) = k+2$. Obviously $y \in (1,\infty)$. Now $g_k(y) > 0$ implies, $\frac{y+k}{y} - \ln(y) > 0$ that is $k > y(k+1)$. This does not hold for all $k \geq 0$. Hence a contradiction. That is there exists some real number in $(1,\infty)$ for which $g_k(x) \leq 0$. Now using Lemmas 3.9 and 3.10 we get that there exist some $c \in (1,\infty)$ or (q,∞) for which the value of g_k is 0. The other two statements hold trivially as g_k is a decreasing function.
2. For $k < 0$, $q = -k$. Hence $g_k(q) = \frac{-k+k}{-k} - \ln(-k) = -\ln(-k)$, which is always negative. As g_k is strictly decreasing in $[q,\infty)$ hence $g_k(x) < 0$ for all $x \in (q,\infty)$.

□

Definition 3.12. For a non-negative integer k, the quantity c denotes the solution to the equation $g_k(c) = 0$. By Lemma 3.11 we know that such a c exists.

Definition 3.13. For an integer k, a positive real p is defined to be,

$$p = \begin{cases} c & \text{if } k \geq 0 \\ q & \text{if } k < 0 \end{cases}$$

Theorem 3.14. *The function f_k is an increasing function in the interval (q,p) and is a decreasing function in the interval (p,∞).*

Proof. From Lemmas 3.5 and 3.6 it is known that the function f_k is differentiable as well as continuous on the interval (q,∞). By using basic rules of differentiation we get $\frac{d}{dx}[f_k(x)] = (x^{1/(x+k)}(\frac{x+k}{x} - \ln(x)))/(x+k)^2$. It can be noticed that this term is positive or negative according to the term $\frac{x+k}{x} - \ln(x)$, as $x^{1/(x+k)}/(x+k)^2$ is always positive. That is the sign of $f_k'(x)$ depends on $g_k(x)$. Now we have the following cases.

Case I: Let $k \geq 0$. Assume two real numbers ε and δ such that $q < \varepsilon < \delta < p$. From Lemmas 3.5 and 3.6 we have, f_k is continuous on $[\varepsilon,\delta]$ and differentiable on (ε,δ). Now for all $x \in (\varepsilon,\delta)$, $x < p = c$. Then by Lemma 3.11, $g_k(x) > 0$. This implies $f_k'(x) > 0$. Hence by Theorem 3.1, f_k is strictly increasing in $[\varepsilon,\delta]$. This holds for all $q < \varepsilon < \delta < p$, so we can say that f_k is strictly increasing in (q,p).

Similarly we can assume real numbers ε and δ such that $p < \varepsilon < \delta < \infty$. Then by Lemma 3.11, for all $x \in (\varepsilon,\delta)$, $f_k'(x) < 0$ as $g_k(x) < 0$. Now applying Theorem 3.1 gives that f_k is strictly decreasing in $[\varepsilon,\delta]$. This holds for all $p < \varepsilon < \delta < \infty$. Hence we can say f_k is decreasing in (q,∞).

Thus, for all $k \geq 0$, f_k is strictly increasing in (q,p) and strictly decreasing in (p,∞).

Case II: Let $k < 0$. Then we have $p = q$. Then the interval $(q,p) = (q,q)$ does not contain any real numbers. Hence we only have to show that f_k is decreasing in $(p,\infty) = (q,\infty)$. By Lemma 3.11, for $k < 0$, $g_k(x) < 0$ for all $x > q$. Now let us assume two real numbers ε and δ such that $q < \varepsilon < \delta < \infty$. Then $f_k'(x) < 0$ for all $x \in (\varepsilon,\delta)$. That is f_k is strictly decreasing in (ε,δ). Again this holds for all $q < \varepsilon < \delta < \infty$. Hence we can say that f_k is strictly decreasing in (q,∞). This completes the second case and hence the proof. □

4 Applications of f_k

In this section, we discuss the application of the function f_k and its properties. To start with, let's explore a couple of problems that can be solved easily using the p value as defined in the last section. (Numerical values shown here and in the following sections, were computed using NumPy 2.1 and Python 3.10, truncated up to the last decimal digit as displayed).

Example 4.1. Find the higher root out of $100000^{1/99998}$ and $100003^{1/100001}$.

Solution. The two roots can be written as $100000^{100000-2} = f_{-2}(100000)$ and $100003^{100003-2} = f_{-2}(100003)$. By 3.14 we know that f_{-2} is strictly decreasing in (p,∞), where $p = 1$. Hence it can be easily concluded that $f_{-2}(100000) > f_{-2}(100003)$, i.e. $100000^{1/99998} > 100003^{1/100001}$. The exact numerical values of the roots are given below for verification.

$$100000^{1/99998} \sim 1.00011513\mathbf{8}$$
$$100003^{1/100001} \sim 1.00011513\mathbf{5}$$

□

Example 4.2. Find the higher root out of $200^{1/217}$ and $201^{1/2018}$.

Solution. Similar to the previous problem, $200^{1/217}$ can be written as $f_{17}(200)$ and $201^{1/2018}$ is equal to $f_{17}(201)$. By 3.14, f_{17} is strictly decreasing in the interval (p,∞). In this case, $p \sim 11.6686$. (See 5). Hence, $200^{1/217} > 201^{1/2018}$.

$$200^{1/217} \sim 1.024\mathbf{7}$$
$$201^{1/218} \sim 1.024\mathbf{6}$$

□

The problems above demonstrate a technique that might be used in general to solve the problem of comparing two roots of the form $a^{1/(a-k)}$ and $b^{1/(b-k)}$. The next step in this direction would be to find a generalization of the same, to be able to compare any two positive real roots of form $a^{1/b}$ and $c^{1/d}$. The following results show that such generalization is possible. The proofs are routine and hence omitted.

Definition 4.3. For two real numbers k and k', let $f_{(k,k')} : (q,q') \to \mathbb{R}$ be a function such that $f_{(k,k')}(x) = x^{1/(x+k+(x-1)k')}$. Here (q,q') is the real interval, where the term $x+k+(x-1)k'$ is positive.

Definition 4.4. For two real numbers k and k', let $g_{(k,k')} : (q,q') \to \mathbb{R}$ is a function such that $g_{(k,k')}(x) = \frac{x+k+(x-1)k'}{x} - (k'+1)\ln(x)$.

Theorem 4.5. *For two distinct positive real roots $a^{1/b}$ and $c^{1/d}$ there exist some k,k' such that $a^{1/b} = f_{(k,k')}(a)$ and $c^{1/d} = f_{(k,k')}(c)$. (k,k') is the solution to the following pair of equations.*

$$a+x+(a-1)\cdot y = b$$
$$c+x+(c-1)\cdot y = d$$

which always exists for $a \neq c$.

Theorem 4.6. *For the function $f_{(k,k')}$, there exists some $p' \in (q,q')$ such that,*

(i) $f_{(k,k')}$ is strictly increasing in the interval (q,p'),
(ii) $f_{(k,k')}$ is strictly decreasing in the interval (p',q').

In fact, p' is either the solution to the equation $g_{(k,k')} = 0$ or $p' = q$.

The last two theorems can now be used to compare any two positive roots $a^{1/b}$ and $c^{1/d}$, whenever both of them lie on the increasing or decreasing side of $f_{(k,k')}$.

Example 4.7. Find the higher root out of $5^{1/18}$ and $7^{1/24}$.

Solution. First, let's find the values k,k' by solving the following pair of equations,

$$5+x+(5-1)\cdot y = 18$$
$$7+x+(7-1)\cdot y = 24$$

This gives $k = 5, k' = 2$. Thus, the roots can now be re-written as, $5^{1/18} = 5^{5+5+(5-1)\cdot 2} = f_{(5,3)}(5)$ and $7^{1/24} = 7^{7+5+(7-1)\cdot 2} = f_{(5,3)}(7)$. Now, for $f_{(5,3)}$ the value of $p' \sim 3.59112$. (See 5) Thus by Theorem 4.6, $5^{1/18} > 7^{1/24}$.

$$5^{1/18} \sim 1.0935$$
$$7^{1/24} \sim 1.0845$$

□

Examples shown in this section, uses the value of p', which can be calculated by solving the equation $g_{(k,k')} = 0$ whenever $p' \neq q'$. If there exists an efficient method of finding p', then the technique of solving root inequalities demonstrated in this section might give better results than conventional methods. We conclude this section, by formally stating this as a problem.

Problem 4.8. *Find an efficient algorithm that can be used to calculate the value of p', as defined in Theorem 4.6, with accuracy of at least n digits after decimal. In particular, such an algorithm will be considered efficient, if it has better time complexity than that of Newton-Raphson method.*

5 Conclusion

Solving root inequalities using the properties of $f_{(k,k')}$ might be useful in systems with limited processing power but enough memory. Embedded systems and IoT devices are a few examples of this. The approach hints at the possibility of building proper algorithms for the applied purpose. Lastly, a solution to Problem 4.8 might put this approach in a more significant position compared to conventional methods of determining roots.

Appendix

k	p
1	3.59112
2	4.31914
3	4.97063
4	5.57239
5	6.13883
6	6.67678
7	7.19322
8	7.69148
9	8.17436
10	8.64403

k	p
11	9.10213
12	9.55001
13	9.98877
14	10.4193
15	10.84237
16	11.25862
17	11.6686
18	12.0728
19	12.47164
20	12.86548

k	p
21	13.25466
22	13.63946
23	14.02015
24	14.39697
25	14.77012
26	15.1398
27	15.50619
28	15.86945
29	16.22973
30	16.58715

k	p
31	16.94186
32	17.29396
33	17.64356
34	17.99076
35	18.33567
36	18.67835
37	19.0189
38	19.35739
39	19.6939
40	20.02849

k	p
41	20.36122
42	20.69216
43	21.02136
44	21.34887
45	21.67475
46	21.99905
47	22.32181
48	22.64307
49	22.96288
50	23.28127

Table 1: p values from 1 to 50.

k	k'	p'
1	1	2.71828
1	2	2.36026
1	3	2.15554
1	4	2.01963
1	5	1.92133
1	6	1.84613
1	7	1.78627
1	8	1.73722
1	9	1.69609
1	10	1.66099

k	k'	p'
2	1	3.18097
2	2	2.71828
2	3	2.45511
2	4	2.28107
2	5	2.15554
2	6	2.05972
2	7	1.98361
2	8	1.92133
2	9	1.86919
2	10	1.82474

k	k'	p'
3	1	3.59112
3	2	3.03396
3	3	2.71828
3	4	2.5101
3	5	2.36026
3	6	2.24609
3	7	2.15554
3	8	2.08153
3	9	2.01963
3	10	1.96691

k	k'	p'
4	1	3.96731
4	2	3.32232
4	3	2.95801
4	4	2.71828
4	5	2.54604
4	6	2.41498
4	7	2.31114
4	8	2.22637
4	9	2.15554
4	10	2.09524

k	k'	p'
5	1	4.31914
5	2	3.59112
5	3	3.18097
5	4	2.91157
5	5	2.71828
5	6	2.57138
5	7	2.45511
5	8	2.36026
5	9	2.28107
5	10	2.2137

Table 2: p' values for $k \in [1,5]$ and $k' \in [1,10]$

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