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Further congruences for $(4,8)$-regular bipartition quadruples modulo powers of 2

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Abstract: We prove some new congruences modulo powers of 2 for $(4,8)$-regular bipartition quadruples, using an algorithmic approach.

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A partition λ of n is a non-negative sequence of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ such that the λ_i's sum up to n. A partition ℓ-tuple of n is an ℓ tuple of partitions $(\Lambda_1, \Lambda_2, \ldots, \Lambda_\ell)$ such that the sum of all the parts of Λ_i is n. Recently, Nayaka [1] introduced (s,t)-regular bipartition quadruples of a positive integer n, denoted by $BQ_{s,t}$ to be the numbers given by the generating function

$$\sum_{n\geq 0} BQ_{s,t}(n)q^n = \frac{(q^s;q^2)_\infty^4 (q^t;q^t)_\infty^4}{(q^s;q^s)_\infty^8},$$

where

$$(a;q)_\infty := \prod_{n\geq 0}(1-aq^n), \quad |q|<1.$$

Nayaka proved several congruence properties satisfied by $BQ_{s,t}(n)$ for different values of (s,t). He proved the results using elementary q-series techniques. The aim of this short note is to extend Nayaka's list of congruences for $(s,t)=(4,8)$ using an algorithmic approach. We use Smoot's [5] implementation of an algorithm of Radu [3] (which we will describe in the next section) to prove this extended list of congruences. This approach has been used very recently by the author [4] to extend some other congruences proved by Nayaka and Naika [2].

In this note, we prove the following result.

Theorem 1. *For all $n \geq 0$, we have*

$$BQ_{4,8}(4n+2) \equiv 0 \pmod{4}, \tag{1}$$
$$BQ_{4,8}(4n+3) \equiv 0 \pmod{64}, \tag{2}$$
$$BQ_{4,8}(8n+4) \equiv 0 \pmod{2}, \tag{3}$$
$$BQ_{4,8}(8n+6) \equiv 0 \pmod{8}, \tag{4}$$
$$BQ_{4,8}(8n+7) \equiv 0 \pmod{256}, \tag{5}$$
$$BQ_{4,8}(16n+9) \equiv 0 \pmod{64}, \tag{6}$$
$$BQ_{4,8}(16n+13) \equiv 0 \pmod{512}, \tag{7}$$
$$BQ_{4,8}(16n+15) \equiv 0 \pmod{512}, \tag{8}$$
$$BQ_{4,8}(32n+17) \equiv 0 \pmod{32}, \tag{9}$$
$$BQ_{4,8}(32n+21) \equiv 0 \pmod{256}, \tag{10}$$
$$BQ_{4,8}(32n+25) \equiv 0 \pmod{1024}, \tag{11}$$
$$BQ_{4,8}(32n+29) \equiv 0 \pmod{1024}, \tag{12}$$
$$BQ_{4,8}(64n+9) \equiv 0 \pmod{64}, \tag{13}$$
$$BQ_{4,8}(64n+33) \equiv 0 \pmod{16}, \tag{14}$$
$$BQ_{4,8}(64n+41) \equiv 0 \pmod{512}, \tag{15}$$
$$BQ_{4,8}(64n+49) \equiv 0 \pmod{64}, \tag{16}$$
$$BQ_{4,8}(64n+57) \equiv 0 \pmod{4096}. \tag{17}$$

Remark 2. *Nayaka [1] had proved the following*

$$BQ_{4,8}(8n+7) \equiv 0 \pmod{128}.$$

Proof of Theorem 1. To prove Theorem 1, we shall use Radu's Ramanujan-Kolberg algorithm [3] as implemented by Smoot [5] for Mathematica, using his package `RaduRK`. Smoot [5] has detailed instructions on its installation and usage. First we invoke the package in Mathematica as follows:

```
In[1] := <<RaduRK'
```

Before running the program, we need to set two global variables q and t:

```
In[2] := {SetVar1[q], SetVar2[t]}
```

The proof of all the congruences are similar, so we shall only prove (5) in details, which can be proved by the procedure call

```
In[1] := RK[4, 8, {-8, 0, 4, 4}, 8, 7].
```

After a few seconds, we get the proof in the form of the following output.

`Out[1] :=`

N:	4
$\{$M,$(r_\delta)_{\delta\mid M}\}$:	$\{8,\{-8,0,4,4\}\}$
m:	8
$P_{m,r}$(j):	$\{7\}$
f_1(q):	$\frac{(q;q)_\infty^{66}\,(q^2;q^2)_\infty^{10}}{q^8\,(q^4;q^4)_\infty^{76}}$
t:	$\frac{(q;q)_\infty^{8}}{q\,(q^4;q^4)_\infty^{8}}$
AB:	$\{1\}$
$\{p_g$(t): g$\in$AB$\}$	$\{21760t^8+23318528t^7+5439488000t^6$ $+517291900928t^5+25120189972480t^4$ $+681697209221120t^3+10484942882471936t^2$ $+85568392920039424t+288230376151711744\}$
Common Factor:	256

The interpretation of this output is as follows.

The first entry in the procedure call `RK[4, 8, {-8, 0, 4, 4}, 8, 7]` corresponds to specifying $N=4$, which fixes the space of modular functions

$$M(\Gamma_0(N)) := \text{the algebra of modular functions for } \Gamma_0(N).$$

The second and third entry of the procedure call `RK[4, 8, {-8, 0, 4, 4}, 8, 7]` gives the assignment $\{M,(r_\delta)_{\delta|M}\}=\{8,(-8,0,4,4)\}$, which corresponds to specifying $(r_\delta)_{\delta|M}=(r_1,r_2,r_4,r_8)=(-8,0,4,4)$, so that

$$\sum_{n\geq 0} BQ_{4,8}(n)q^n=\prod_{\delta|M}(q^\delta;q^\delta)_\infty^{r_\delta}=\frac{(q^4;q^4)^4(q^8;q^8)^4}{(q;q)^8}.$$

The last two entries of the procedure call `RK[4, 8, {-8, 0, 4, 4}, 8, 7]` corresponds to the assignment $m=8$ and $j=7$, which means that we want the generating function

$$\sum_{n\geq 0} BQ_{4,8}(mn+j)q^n=\sum_{n\geq 0} BQ_{4,8}(8n+7)q^n.$$

So, $P_{m,r}(j)=P_{8,r}(7)$ with $r=(-8,0,4,4)$.

The output $P_{m,r}(j):=P_{8,(-8,0,4,4)}(7)=\{7\}$ means that there exists an infinite product

$$f_1(q)=\frac{(q;q)_\infty^{66}\,(q^2;q^2)_\infty^{10}}{q^8\,(q^4;q^4)_\infty^{76}},$$

such that

$$f_1(q)\sum_{n\geq 0} BQ_{4,8}(8n+7)q^n\in M(\Gamma_0(4)).$$

Finally, the output

$$t = \frac{(q;q)_\infty^8}{q\,(q^4;q^4)^8_\infty}, \quad AB = \{1\}, \quad \text{and} \quad \{p_g(\text{t}) \colon \text{g} \in AB\},$$

presents a solution to the question of finding a modular function $t \in M(\Gamma_0(4))$ and polynomials $p_g(t)$ such that

$$f_1(q)\sum_{n\geq 0} BQ_{4,8}(8n+7)q^n = \sum_{g\in AB} p_g(t)\cdot g$$

In this specific case, we see that the singleton entry in the set $\{p_g(\text{t}) \colon \text{g} \in AB\}$ has the common factor 256, thus proving equation (5).

The other congruences in Theorem 1 can be proved in a similar way. For instance, to prove (17) we run the procedure call `RK[4, 8, {-8, 0, 4, 4}, 64, 57]`. The output file generated by Mathematica which proves all the congruences in Theorem 1 can be downloaded from https://manjilsaikia.in/publ/mathematica/BQ-4-8.nb. □

For more details on the steps described above, one can consult Radu [3] and Smoot [5].

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