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Formal triangular matrix ring with nil clean index 4

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Abstract: For an element $a \in R$, where R is an associative ring with unity. Let $\eta(a) = \{e \in R \mid e^2 = e \text{ and } a - e \in \text{nil}(R)\}$. The nil clean index of R, denoted by $\text{Nin}(R)$, is defined by [1]

$$\text{Nin}(R) = \sup\{\mid \eta(a) \mid : a \in R\}.$$

In this article we have characterized formal triangular ring $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ with nil clean index 4.

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1 Introduction

Throughout this article R denotes a associative ring with unity. The set of nilpotents and set of idempotents are denoted by $\text{nil}(R)$ and $\text{idem}(R)$ respectively. The cyclic group of order n is denoted by C_n and $|S|$ denotes the cardinality of the set S. For an element $a \in R$, if $a - e \in \text{nil}(R)$ for some $e \in \text{idem}(R)$, then $a = e + (a - e)$ is said to be a nil clean expression of a in R and a is called a nil clean element [3, 2]. The ring R is called nil clean if each of its elements is nil clean.

For an element $a \in R$, let $\eta(a) = \{e \in R \mid e^2 = e \text{ and } a - e \in \text{nil}(R)\}$. The nil clean index of R, denoted by $\text{Nin}(R)$, is defined by $\text{Nin}(R) = \sup\{\mid \eta(a) \mid : a \in R\}$[1]. Characterization of arbitrary ring with nil clean indices 1, 2 and certain sufficient conditions for a ring to be of nil clean index 3 are given in [1]. In this article we have characterized formal triangular ring $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ with nil clean index 4, here A and B are rings and M is a $A - B-$bimodule. Following results about nil clean index will be used in this article.

Lemma 1.1 (1, Lemma 2.5)**.** *Let* $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$*, where A and B are rings, and* ${}_AM_B$ *is a bimodule. Let* $Nin(A) = n$ *and* $Nin(B) = m$*. Then*

1. $Nin(R) \geq |M|$.
2. *If* $(M, +) \cong C_{p^k}$*, where p is a prime and* $k \geq 1$*, then* $Nin(R) \geq n + \lceil \frac{n}{2} \rceil (|M| - 1)$*, where* $\lceil \frac{n}{2} \rceil$ *denotes the least integer greater than or equal to* $\frac{n}{2}$.

3. *Either* $Nin(R) \geq nm + |M| - 1$ *or* $Nin(R) \geq 2nm$.

Lemma 1.2 (1, Lemma 2.6). *Let* $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, *where A and B are rings,* ${}_AM_B$ *is a bimodule with* $(M,+) \cong C_{2^r}$. *Then* $Nin(R) = 2^r Nin(A)Nin(B)$.

Theorem 1.3 (1, Theorem 4.1). $Nin(R) = 2$ *if and only if* $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, *where* $Nin(A) = Nin(B) = 1$ *and* ${}_AM_B$ *is a bimodule with* $|M| = 2$.

Theorem 1.4 (1, Proposition 4.2). *If* $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, *where* $Nin(A) = Nin(B) = 1$ *and* ${}_AM_B$ *is a bimodule with* $|M| = 3$ *then* $Nin(R) = 3$.

2 Main result

Theorem 2.1. *Let* $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, *where A and B are rings,* ${}_AM_B$ *is a non trivial bimodule. Then* $\mathrm{Nin}(R) = 4$ *if and only if one of the following holds:*

(1) $(M,+) \cong C_2$ *and* $\mathrm{Nin}(A)\,\mathrm{Nin}(B) = 2$.

(2) $(M,+) \cong C_4$ *and* $\mathrm{Nin}(A) = \mathrm{Nin}(B) = 1$.

(3) $(M,+) \cong C_2 \oplus C_2$ *and one of the following*

- (a) $\mathrm{Nin}(A) = \mathrm{Nin}(B) = 1$.
- (b) $\mathrm{Nin}(A) = 1,\ B = \begin{pmatrix} S & W \\ 0 & T \end{pmatrix}$, *Where* $\mathrm{Nin}(S) = \mathrm{Nin}(T) = 1$ *and* $|W| = 2$, *and* $eM(1_B - f) + (1_A - e)Mf \neq 0$ *for all* $e^2 = e \in A$ *and* $f \in \eta(b)$, *where* $b \in B$ *with* $|\eta(b)| = 2$.
- (c) $\mathrm{Nin}(B) = 1,\ A = \begin{pmatrix} S & W \\ 0 & T \end{pmatrix}$, *Where* $\mathrm{Nin}(S) = \mathrm{Nin}(T) = 1$ *and* $|W| = 2$, *and* $eM(1_B - f) + (1_A - e)Mf \neq 0$ *for all* $e^2 = e \in B$ *and* $f \in \eta(a)$, *where* $a \in A$ *with* $|\eta(a)| = 2$.

Proof : $(\Leftarrow)$ If (1) holds then by **Lemma 1.2**, we get $\mathrm{Nin}(R) = 4$.
If (2) holds then $\mathrm{Nin}(R) \geq |M| = 4$, Now, for any $\alpha = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in R$.

$$\eta(\alpha) = \left\{ \begin{pmatrix} e & w \\ 0 & f \end{pmatrix} \in R :\ e \in \eta(a),\ f \in \eta(b),\ w = ew + fw \right\}.$$

Because $|M| = 4,\ |\eta(a)| \leq 1$ and $|\eta(b)| \leq 1$, it follows that $|\eta(\alpha)| \leq 4$. Hence $\mathrm{Nin}(R) = 4$.

(3) (a) Similar to case (2).

Proof of $3(b)$ and $3(c)$ are identical, hence proof is given only for $3(c)$. Suppose $(3)(c)$ holds, clearly $\mathrm{Nin}(R) \geq |M| = 4$. Let $\alpha = \begin{pmatrix} a & w \\ 0 & b \end{pmatrix} \in R$. We show that $|\eta(\alpha)| \leq 4$ and hence $\mathrm{Nin}(R) = 4$ holds. Since $\mathrm{Nin}(B) = 1$, we can assume that $\eta(b) = \{f_0\}$. Then as above we have

$$\eta(\alpha) = \left\{ \begin{pmatrix} e & z \\ 0 & f_0 \end{pmatrix} \in R : e \in \eta(a), z = ez + zf_0 \right\}.$$

If $|\eta(a)| \leq 1$, then $|\eta(\alpha)| \leq |\eta(a)|.|M| \leq 4$. So we can assume that $|\eta(a)| = 2$. Write $\eta(a) = \{e_1,\ e_2\}$. Thus $\eta(\alpha) = T_1 \bigcup T_2$, where

$$T_i = \left\{ \begin{pmatrix} e_i & z \\ 0 & f_0 \end{pmatrix} \in R : (1_A - e_i)z = zf_0 \right\} \qquad (i = 1,\ 2).$$

Since $\eta(1_A - a) = \{1_A - e_1,\ 1_A - e_2\}$, the assumption $(3)(c)$ shows that $\{z \in M : (1_A - e_i)z = zf_0\}$ is a proper subgroup of $(M,+)$; so $|T_i| \leq 2$ for $i = 1,\ 2$. Hence $|\eta(\alpha)| = |T_1|.|T_2| \leq 4$.

($\Rightarrow$) Suppose $\mathrm{Nin}(R) = 4$. Then $2 \leq |M| \leq \mathrm{Nin}(R) = 4$. If $|M| = 2$ then $\mathrm{Nin}(A)\,\mathrm{Nin}(B) = 2$ by Lemma 1.2, so (1) holds.

Suppose $|M| = 3$, then we have by **Lemma 1.1** $\mathrm{Nin}(A) + |M| - 1 \leq \mathrm{Nin}(R)$, showing $\mathrm{Nin}(A) \leq 2$, similarly $\mathrm{Nin}(B) \leq 2$. But $\mathrm{Nin}(A) = 2 = \mathrm{Nin}(B)$ will give $\mathrm{Nin}(R) \geq 6$ by **Lemma 1.1** and $\mathrm{Nin}(A) = \mathrm{Nin}(B) = 1$ will give $\mathrm{Nin}(R) = 3$ by **Theorem 1.4**. Hence last possibility is $\mathrm{Nin}(A)\,\mathrm{Nin}(B) = 2$, so without loss of generality we assume that $\mathrm{Nin}(A) = 2$ and $\mathrm{Nin}(B) = 1$. Write $(M,+) = \{0, x, 2x\}$. Now by **Theorem 1.3** we have $A = \begin{pmatrix} T & N \\ 0 & S \end{pmatrix}$, where T & S are rings, ${}_TN_S$ is bimodule with $\mathrm{Nin}(T) = \mathrm{Nin}(S) = 1$ and $|N| = 2$. Note that for $e \in \mathrm{idem}(A)$, $ex \in \{0,\ x\}$, $x \in M$, as if $ex = 2x$, we have $2x = ex = e(ex) = e(2x) = e(x+x) = ex + ex = 2x + 2x = 4x = x$ which is not true.

Now Let $a = \begin{pmatrix} 1_T & 0 \\ 0 & 0 \end{pmatrix} \in A$ such that
$a = \begin{pmatrix} 1_T & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1_T & y \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -y \\ 0 & 0 \end{pmatrix}$. Let us denote,

$$e_1 = \begin{pmatrix} 1_T & 0 \\ 0 & 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 1_T & y \\ 0 & 0 \end{pmatrix},\ n_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad n_2 = \begin{pmatrix} 0 & -y \\ 0 & 0 \end{pmatrix},$$

where $e_1,\ e_2 \in \mathrm{idem}(A)$ & $n_1,\ n_2 \in \mathrm{nil}(R)$. Now we have following cases:

Case I: Let $e_1x = e_2x = 0$ then we have an element
$\beta = \begin{pmatrix} (1_A - a) & 0 \\ 0 & 0 \end{pmatrix} \in R$ such that

$$\begin{aligned} \beta &= \begin{pmatrix} (1_A - e_1) & z \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -n_1 & -x \\ 0 & 0 \end{pmatrix} && \forall z \in M \\ &= \begin{pmatrix} (1_A - e_2) & z \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -n_2 & -x \\ 0 & 0 \end{pmatrix} && \forall z \in M \end{aligned}$$

are six nil clean expressions for β, which implies $|\eta(\beta)| \geq 6$, that is $\mathrm{Nin}(R) \geq 6$, which is not possible.

Case II: Let $e_1x = e_2x = x$ then we have an element
$\alpha = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in R$ such that

$$\begin{aligned} \alpha &= \begin{pmatrix} e_1 & z \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} n_1 & -x \\ 0 & 0 \end{pmatrix} && \forall z \in M \\ &= \begin{pmatrix} e_2 & z \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} n_2 & -x \\ 0 & 0 \end{pmatrix} && \forall z \in M \end{aligned}$$

are six nil clean expressions for α, which implies $|\eta(\alpha)| \geq 6$, that is $\mathrm{Nin}(R) \geq 6$, which is also not possible.

Case III: Let $e_1x = x$ and $e_2x = 0$, then we have $e_1 - e_2)x = x$. Let $j = e_1 - e_2$, then clearly $j \in \mathrm{nil}(A)$, we have $jx = x \ \Rightarrow \ (1_A - j)x = 0 \ \Rightarrow \ x = 0$, (as $(1 - j)$ is an unit in A), which is not possible.

Case IV: Let $e_1x = 0$ and $e_2x = x$, as in case III, we get a contradiction. Hence if $M \cong C_3$, $\mathrm{Nin}(R)$ is never 3.
Suppose $|M| = 4$. If $(M,+) \cong C_4$, then $\mathrm{Nin}(A)\,\mathrm{Nin}(B) = 1$ by Lemma 1.2. So (2) holds. Let $(M,+) \cong C_2 \oplus C_2$. Since $\mathrm{Nin}(R) = 4$, by Lemma 1.1 we have $\mathrm{Nin}(A)\,\mathrm{Nin}(B) \leq 2$. If $\mathrm{Nin}(A)\,\mathrm{Nin}(B) = 1$ then $(3)(a)$ holds. If $\mathrm{Nin}(A)\,\mathrm{Nin}(B) = 2$, without loss of generality we can assume $\mathrm{Nin}(A) = 2$ and $\mathrm{Nin}(B) = 1$. So by **Theorem 1.3**, we have $A = \begin{pmatrix} S & W \\ 0 & T \end{pmatrix}$ where $\mathrm{Nin}(S) = \mathrm{Nin}(T) = 1$, and $|W| = 2$. To complete the proof suppose in contrary that $eM(1_B - f) + (1_A - e)Mf = 0$ for some $f^2 = f \in B$ and $e \in \eta(a)$, where $a \in A$ with $|\eta(a)| = 2$. Then

$ew = wf$ for all $w \in M$. It is easy to check that $\eta(a) = \{e,\ e+j\}$ where $j = \begin{pmatrix} 0 & w_0 \\ 0 & 0 \end{pmatrix} \in A$ with $0 \neq w_0 \in W$.
Thus, for $\gamma := \begin{pmatrix} 1_A - e & 0 \\ 0 & f \end{pmatrix}$,

$$\eta(\gamma) \supseteq \left\{ \begin{pmatrix} 1_A - e & 0 \\ 0 & f \end{pmatrix}, \begin{pmatrix} 1_A - (e+j) & 0 \\ 0 & f \end{pmatrix}, \begin{pmatrix} 1_A - e & w \\ 0 & f \end{pmatrix} : w \in M \right\}.$$

So $|\eta(\gamma)| \geq 5$, a contradiction. So $(3)(c)$ holds, similarly $(3)(b)$ can be proved. □

Acknowledgments

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