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On the number of partitions of n whose product of the summands is at most n

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Abstract: We prove an explicit formula to count the partitions of n whose product of the summands is at most n. In the process, we also deduce a result to count the multiplicative partitions of n.

সাৰাংশ: n ৰ যিসমূহ বিভাজনৰ অংশসমূহৰ পূৰণফল অতি বেছি n হয়, তাৰ মুঠ সংখ্যা গণনা কৰিব পৰা এটা স্পষ্ট সূত্ৰ আমি প্ৰমাণ কৰোঁ। এই প্ৰক্ৰিয়াটোত আমি n ৰ গুণকীয় বিভাজনসমূহ গণনা কৰিব পৰা এটা ফলাফলো নিৰ্ণয় কৰোঁ।

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1 Introduction

A partition of a non-negative integer n is a representation of n as a sum of unordered positive integers which are called parts or summands of that partition. The number of partitions of n is denoted by $p(n)$. For example, the partitions of 7 are:

$$\begin{aligned}&1+1+1+1+1+1+1,\\&1+1+1+1+1+2, 1+1+1+1+3, 1+1+1+4, 1+1+5, 1+6, 7,\\&1+1+1+2+2, 1+1+2+3, 1+2+4, 2+5, 1+3+3, 3+4,\\&1+2+2+2, 2+2+3.\end{aligned}$$

So, $p(7) = 15$.

Many restricted partitions such as partitions with only odd summands, partitions with distinct summands, partitions whose summands are divisible by a certain number, partitions with restricted number of summands, partitions with designated summands, etc. have been studied over the last two and half centuries. The partitions of n whose product of the summands is at most n are another kind of restricted partition. We denote the total number of such partitions of n by $p_{\leq n}(n)$. Also, we denote by $p_{=n}(n)$ the number of partitions of n whose product of the summands is equal to n. Similarly we use the notations $p_{<n}(n)$, $p_{\geq n}(n)$, etc.

The value of the product of the summands of a partition depends on the summands of that partition which are greater than one. We call these summands the non-one summands or non-one parts. Let n be a positive integer and the canonical decomposition of n as a product of distinct primes be

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r},$$

where $\alpha_i \in \mathbb{N}$ for all $i = 1, 2, \ldots, r$.

As a consequence of the fundamental theorem of arithmetic, we have the following proposition.

Proposition 1.1. *The product of the summands of a partition of n is equal to n if and only if*

1. *each summand is a divisor of n, and*
2. *p_i appears as a factor of the summands exactly α_i times, for all $i = 1, 2, \ldots, r$.*

Proposition 1.2. *Let the product of the summands of a partition of n be n, and each non-one summand of the partition has one prime factor. The total number of such partitions of n is*

$$\prod_{i=1}^{r} p(\alpha_i).$$

Proof. Let $\lambda_1 + \lambda_2 + \cdots + \lambda_t$ be a partition of α_1. Then,

$$p_1^{\lambda_1} + p_1^{\lambda_2} + \cdots + p_1^{\lambda_t} \leq p_1^{\alpha_1},$$

since, for all positive integers $a, b > 1$, we have $a + b \leq ab$ and the equality holds only for $a = b = 2$. So, for any positive integer $c > 1$, we have $a + b + c \leq ab + c < abc$, and so on.

Here, each non-one summand of a partition of n has only one prime factor. So, by the Proposition 1.1, we get the result. □

A multiplicative partition of a positive integer n is a representation of n as a product of unordered positive non-one integers. The multiplicative partition function was introduced by MacMahon [6], [7] in 1923. Since then many properties of this function have been studied. Some works can be found in [1], [2], [3], [4], [5] and [8]. $p_{=n}(n)$ is equal to the total number of multiplicative partitions of n. In this paper, we introduce and prove formulas for $p_{\leq n}(n)$ and $p_{<n}(n)$, and using both we find $p_{=n}(n)$.

2 Formulas for $p_{\leq n}(n)$, $p_{<n}(n)$ and $p_{=n}(n)$

The floor function of x, denoted by $\lfloor x \rfloor$, is the greatest integer less than or equal to x, where x is any real number. In this section this function appears many times.

Theorem 2.1. *We have*

$$p_{\leq n}(n) = n + \sum_{k=2}^{\ell} \sum_{i_1=2}^{\lfloor \sqrt[k]{n} \rfloor} \sum_{i_2=i_1}^{\left\lfloor \sqrt[k-1]{\frac{n}{i_1}} \right\rfloor} \sum_{i_3=i_2}^{\left\lfloor \sqrt[k-2]{\frac{n}{i_1 i_2}} \right\rfloor} \cdots \sum_{i_{k-1}=i_{k-2}}^{\left\lfloor \sqrt{\frac{n}{i_1 i_2 \cdots i_{k-2}}} \right\rfloor} \left(\left\lfloor \frac{n}{i_1 i_2 \cdots i_{k-1}} \right\rfloor - i_{k-1} + 1 \right), \tag{2.1}$$

where $2^{\ell} \leq n < 2^{\ell+1}$.

Proof. $1 + 1 + 1 + \cdots + 1$ is the only partition of n where there is no non-one summand. The partitions of n where there is only one non-one summand are

$$1 + \cdots + 1 + 1 + 2, 1 + \cdots + 1 + 3, \ldots, 1 + 1 + (n-2), 1 + (n-1) \text{ and } n.$$

So we get $n-1$ partitions whose product of summands $< n$ and one partition whose product of summands $= n$.

Now, we count the partitions where there are only two non-one summands.

$$\begin{aligned}
1+\cdots+1+1+2+2, 1+\cdots+1+2+3, 1+\cdots+2+4, \ldots, 1+2+(n-3), 2+(n-2),\\
1+\cdots+1+3+3, 1+\cdots+3+4, \ldots, 1+3+(n-4), 3+(n-3),\\
\ldots\ldots,\\
\ldots\ldots,\\
\text{and so on.}
\end{aligned}$$

Here, in the first row, where the initial non-one summand is 2, the total number of partitions whose product of summands $\leq n$ is $\lfloor \frac{n}{2} \rfloor - 1$. That of second row is $\lfloor \frac{n}{3} \rfloor - 2$. In this way we can count up to the row where initial non-one summand is $\lfloor \sqrt{n} \rfloor$; since $(\lfloor \sqrt{n} \rfloor + 1)(\lfloor \sqrt{n} \rfloor + 1) > n$. Thus, the total number of such partitions having exactly two non-one summands is

$$\sum_{i_1=2}^{\lfloor \sqrt{n} \rfloor} \left(\left\lfloor \frac{n}{i_1} \right\rfloor - i_1 + 1 \right).$$

To see the clear picture, we now count such partitions where there are four non-one summands.

$$\begin{aligned}
1+\cdots+1+1+2+2+2+2, 1+\cdots+1+2+2+2+3, 1+\cdots+2+2+2+4, \ldots, 2+2+2+(n-6),\\
1+\cdots+1+2+2+3+3, 1+\cdots+2+2+3+4, \ldots, 2+2+3+(n-7),\\
1+\cdots+1+2+2+4+4, \ldots, 2+2+4+(n-8),\\
\ldots\ldots,\\
\ldots\ldots,\\
1+\cdots+1+2+3+3+3, 1+\cdots+2+3+3+4, 1+\cdots+2+3+3+5, \ldots, 2+3+3+(n-8),\\
1+\cdots+2+3+4+4, 1+\cdots+2+3+4+5, \ldots, 2+3+4+(n-9),\\
1+\cdots+2+3+5+5, \ldots, 2+3+5+(n-10),\\
\ldots\ldots,\\
\ldots\ldots,\\
1+\cdots+1+3+3+3+3, 1+\cdots+3+3+3+4, 1+\cdots+3+3+3+5, \ldots, 3+3+3+(n-9),\\
\ldots\ldots,\\
\ldots\ldots,\\
\text{and so on.}
\end{aligned}$$

In the first row, the number of partitions whose product of summands $\leq n$ is $\lfloor \frac{n}{2\cdot 2\cdot 2} \rfloor - 1$. So, in the first group of rows, the number of partitions whose product of summands $\leq n$ is

$$\sum_{i_3=2}^{\lfloor \sqrt{\frac{n}{2\cdot 2}} \rfloor} \left(\left\lfloor \frac{n}{2\cdot 2\cdot i_3} \right\rfloor - i_3 + 1 \right).$$

In the second group of rows it equals to

$$\sum_{i_3=3}^{\lfloor \sqrt{\frac{n}{2\cdot 3}} \rfloor} \left(\left\lfloor \frac{n}{2\cdot 3\cdot i_3} \right\rfloor - i_3 + 1 \right).$$

When the initial non-one summand is 2, then we can count up to the partitions where the second non-one summand is $\left\lfloor \sqrt[3]{\frac{n}{2}} \right\rfloor$. Thus, when the initial non-one summand is 2, then the number of partitions whose product of summands $\leq n$ is

$$\sum_{i_2=2}^{\left\lfloor \sqrt[3]{\frac{n}{2}} \right\rfloor} \sum_{i_3=i_2}^{\left\lfloor \sqrt{\frac{n}{2\cdot i_2}} \right\rfloor} \left(\left\lfloor \frac{n}{2\cdot i_2 \cdot i_3} \right\rfloor - i_3 + 1 \right).$$

In this way, we can count up to the group of rows, where the initial non-one summand is $\lfloor \sqrt[4]{n} \rfloor$. Therefore, the total number of such partitions having exactly four non-one summands is

$$\sum_{i_1=2}^{\lfloor \sqrt[4]{n} \rfloor} \sum_{i_2=i_1}^{\left\lfloor \sqrt[3]{\frac{n}{i_1}} \right\rfloor} \sum_{i_3=i_2}^{\left\lfloor \sqrt{\frac{n}{i_1\cdot i_2}} \right\rfloor} \left(\left\lfloor \frac{n}{i_1\cdot i_2 \cdot i_3} \right\rfloor - i_3 + 1 \right).$$

Thus, when there are exactly k non-one summands, then the number of partitions whose product of summands $\leq n$ is

$$\sum_{i_1=2}^{\lfloor \sqrt[k]{n} \rfloor} \sum_{i_2=i_1}^{\left\lfloor \sqrt[k-1]{\frac{n}{i_1}} \right\rfloor} \sum_{i_3=i_2}^{\left\lfloor \sqrt[k-2]{\frac{n}{i_1 i_2}} \right\rfloor} \cdots \sum_{i_{k-1}=i_{k-2}}^{\left\lfloor \sqrt{\frac{n}{i_1 i_2 \cdots i_{k-2}}} \right\rfloor} \left(\left\lfloor \frac{n}{i_1 i_2 \cdots i_{k-1}} \right\rfloor - i_{k-1} + 1 \right). \tag{2.2}$$

If $2^\ell \leq n < 2^{\ell+1}$, then a partition of n, whose product of summands is $\leq n$, must have at most ℓ non-one summands. This completes the proof. □

Corollary 2.2. *We have*

$$p_{\leq n}(n) = p_{<n+1}(n+1).$$

Proof. If we observe the expression (2.2), then we see that if the product of the summands is n then n is divisible by $i_1 i_2 \cdots i_{k-1}$. So, to find $p_{<n}(n)$, we can take

$$\left\lfloor \frac{n-1}{i_1 i_2 \cdots i_{k-1}} \right\rfloor - i_{k-1} + 1,$$

since $a \nmid n$ implies $\left\lfloor \frac{n}{a} \right\rfloor = \left\lfloor \frac{n-1}{a} \right\rfloor$ for all positive integer a.

Again, when there are exactly k non-one summands, then the product of the summands is n if $\sqrt[k]{n}$ is an integer. Also, if $\sqrt[k]{n}$ is not an integer, then $\lfloor \sqrt[k]{n} \rfloor = \lfloor \sqrt[k]{n-1} \rfloor$. So, to find $p_{<n}(n)$, we can take the value of i_1 up to $\lfloor \sqrt[k]{n-1} \rfloor$. In this way we get, when there are exactly k non-one summands, then the number of partitions whose product of summands $< n$ is

$$\sum_{i_1=2}^{\lfloor \sqrt[k]{n-1} \rfloor} \sum_{i_2=i_1}^{\left\lfloor \sqrt[k-1]{\frac{n-1}{i_1}} \right\rfloor} \sum_{i_3=i_2}^{\left\lfloor \sqrt[k-2]{\frac{n-1}{i_1 i_2}} \right\rfloor} \cdots \sum_{i_{k-1}=i_{k-2}}^{\left\lfloor \sqrt{\frac{n-1}{i_1 i_2 \cdots i_{k-2}}} \right\rfloor} \left(\left\lfloor \frac{n-1}{i_1 i_2 \cdots i_{k-1}} \right\rfloor - i_{k-1} + 1 \right).$$

Now, we get the following two cases.

1. $2^\ell < n < 2^{\ell+1}$. In this case, $2^\ell \leq n-1 < 2^{\ell+1}$ and a partition of n whose product of summands is $< n$ must have at most ℓ non-one summands.

2. $n = 2^\ell$. In this case, $2^{\ell-1} \leq n-1 < 2^\ell$ and we must have at most $\ell - 1$ non-one summands.

So, combining both cases, we can take the value of k up to s, where $2^s \leq n-1 < 2^{s+1}$. Hence,

$$\begin{aligned} p_{<n}(n) &= n-1+\sum_{k=2}^{s}\sum_{i_1=2}^{\left\lfloor\sqrt[k]{n-1}\right\rfloor}\sum_{i_2=i_1}^{\left\lfloor\sqrt[k-1]{\frac{n-1}{i_1}}\right\rfloor}\sum_{i_3=i_2}^{\left\lfloor\sqrt[k-2]{\frac{n-1}{i_1 i_2}}\right\rfloor}\cdots\sum_{i_{k-1}=i_{k-2}}^{\left\lfloor\sqrt{\frac{n-1}{i_1 i_2\cdots i_{k-2}}}\right\rfloor}\left(\left\lfloor\frac{n-1}{i_1 i_2\cdots i_{k-1}}\right\rfloor - i_{k-1}+1\right) \\ &= p_{\leq n-1}(n-1). \end{aligned}$$

□

Corollary 2.3. *For $n > 1$, we have*

$$p_{=n}(n) = p_{\leq n}(n) - p_{\leq n-1}(n-1).$$

Corollary 2.4. *If n is a prime, then*

$$p_{\leq n}(n) = p_{\leq n-1}(n-1)+1.$$

Proof. If n is a prime, then the product of the summands of any partition of n with more than one non-one summands can not be n. So, from the proof of the Corollary 2.2, we get

$$\begin{aligned} p_{\leq n}(n) &= 1+n-1+\sum_{k=2}^{s}\sum_{i_1=2}^{\left\lfloor\sqrt[k]{n-1}\right\rfloor}\sum_{i_2=i_1}^{\left\lfloor\sqrt[k-1]{\frac{n-1}{i_1}}\right\rfloor}\sum_{i_3=i_2}^{\left\lfloor\sqrt[k-2]{\frac{n-1}{i_1 i_2}}\right\rfloor}\cdots\sum_{i_{k-1}=i_{k-2}}^{\left\lfloor\sqrt{\frac{n-1}{i_1 i_2\cdots i_{k-2}}}\right\rfloor}\left(\left\lfloor\frac{n-1}{i_1 i_2\cdots i_{k-1}}\right\rfloor - i_{k-1}+1\right), \\ &\qquad \text{where } 2^s \leq n-1 < 2^{s+1} \\ &= 1+p_{\leq n-1}(n-1). \end{aligned}$$

□

3 Concluding remarks

A few values of $p_{=n}(n)$, $p_{\leq n}(n)$, $p_{\geq n}(n)$ and $p_{>n}(n)$ can be found in the OEIS sequences A001055, A096276, A319005 and A114324 respectively. The Corollary 2.2 implies that the sequence A096276 gives the values of $p_{<n}(n)$ too.

It seems that an explicit formula for $p_{\geq n}(n)$ or $p_{>n}(n)$ in the spirit of Theorem 2.1 can be found. We leave this as an open problem.

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