JOURNAL OF THE ASSAM ACADEMY OF MATHEMATICS
Journal homepage: jaam.aamonline.org.in

Conjectures on congruences of binomial coefficients modulo higher powers of a prime number

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Received 22 July 2022; Revised 26 August 2023; Accepted 06 October 2023; Published 31 December 2023

Abstract: Some congruences modulo p^3 and p^2, for a prime p involving binomial coefficients are stated. These appear to be novel.

AMS Classification: 11A07, 11A41.

Key words and phrases: binomial coefficients, congruences.

A few years ago, the author and Vogrinc proved [4, 5] that for a prime number p and $n > p$, the following holds

$$\binom{n}{p} \equiv \left\lfloor \frac{n}{p} \right\rfloor \pmod{p}, \tag{1}$$

along with several other results on the periodicity of such a sequence. In fact, the following more stronger statement was proved by the author and Laugier [3]: for a prime p and natural numbers $n > p$ and $k \geq 1$, the following holds

$$\binom{n}{p^k} \equiv \left\lfloor \frac{n}{p^k} \right\rfloor \pmod{p}. \tag{2}$$

The periodicity of binomial coefficients is a fascinating object of study, and has been studied extensively (see the references in [4, 5, 3] for example). Some other work that the author and Laugier did in this direction can be found in the unpublished manuscript [2].

What the author and his collaborators did not study were relations for higher powers of p involving binomial coefficients and floor functions; so here in this note we make the following conjectures.

Conjecture 1. *For a prime $p > 3$ and natural number $n \geq 1$, the following is true*

$$\binom{pn-1}{p} \equiv n-1 \pmod{p^3}.$$

Conjecture 2. *We have for all $n \geq 1$,*

$$\binom{3^\ell n-k}{n} \equiv \left\lfloor \frac{3^\ell n-k}{3} \right\rfloor \pmod{3^2},$$

where $k \equiv 0, 1 \pmod{3}$ and $\ell \geq 1$.

Some remarks are in order:

1. Conjecture 1 has been verified upto $n = 1000$ and $p = 19$. Conjecture 2 has been verified upto $n = 1000$.

2. The congruences (1) and (2) were obtained using elementary techniques and relied on the famous Lucas' identity. Congruence (1) also has a combinatorial proof. Such approaches are unlikely to work in the case of Conjectures 1 and 2 due to presence of higher power of p in the moduli.

3. Bailey [1] proved a related result
$$\binom{pn}{p} \equiv n \pmod{p^3},$$
although his techniques do not seem to work for our cases. In fact, he proved a much more general result in his paper which we do not state here.

4. It seems that Conjecture 2 can be extended further. Is the following true?
$$\binom{p^\ell n-k}{n} \equiv \left\lfloor \frac{p^\ell n-k}{p} \right\rfloor \pmod{p^3},$$
for $k \equiv 0,1 \pmod{p}$, $\ell \geq 1$ (and for possibly other values of k).

5. Even more complicated supercongruences can be conjectured, for higher powers of p, but with different floor functions. Such congruences for p^3 are well studied, but not with the floor function, as is the case here.

Note added in proof. Daniel Yaqubi has informed the author that he has proved the conjectures in the paper, but the author has not yet seen the proofs.

Acknowledgements

A part of this work was done when the author was a doctoral student at the University of Vienna, where he was supported by the Austrian Science Foundation FWF, START grant Y463. Another part of this work was done at Cardiff and is partially supported by the Leverhulme Trust Research Project Grant RPG-2019-083. The author thanks the referee for helpful comments.

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