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Representing Even Perfect and Near-Perfect Numbers as Sums of Cubes

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Abstract: Motivated by recent results of Farhi and Ulas we show that for each $n \equiv \pm 1 \pmod 6$, the Diophantine equation $2^{n-1}(2^n - 1) = x^3 + y^3 + z^3$ has at least three solutions. We also prove results about the representation of even near-perfect numbers with two distinct prime factors as sums of integral cubes.

সাৰাংশ: ফাৰ্হি আৰু উলাছৰ শেহতীয়া ফলাফলৰ দ্বাৰা অনুপ্ৰাণিত হৈ আমি দেখুৱাইছোঁ যে প্ৰতিটো $n \equiv \pm 1 \pmod 6$ ৰ বাবে, $2^{n-1}(2^n - 1) = x^3 + y^3 + z^3$ ডায়ফেণ্টাইন সমীকৰণটোৰ অতি কমেও তিনিটা সমাধান থাকে। দুটা পৃথক মৌলিক উৎপাদক যুক্ত যুগ্ম নিকট-নিখুঁত সংখ্যাক অখণ্ড ঘনকৰ যোগফল ৰূপে প্ৰকাশ কৰা সম্পৰ্কীয় ফলাফলৰ প্ৰমাণো আমি আগবঢ়াইছোঁ।

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1 Introduction

One of the most frequently studied class of numbers in number theory are the *perfect numbers*. A number n is called a perfect number if it is equal to the sum of its proper divisors. More succinctly, if $\sigma(n) = 2n$ then n is called a perfect number, where $\sigma(n)$ denotes the sum of the distinct positive divisors of n. Even perfect numbers have been known since antiquity and they can be characterized completely by the work of Euclid and Euler (see for instance the book by Hardy and Wright [2]). In particular, any even perfect number n can be written as $n := P_p = 2^{p-1}(2^p - 1)$, where p and $2^p - 1$ are both primes. However, to date no odd perfect number has been found. In fact, it is not known whether they exist or not.

Motivated by this search for odd perfect numbers, various generalizations of perfect numbers have been studied extensively in the literature (for instance, see the joint work of the third author with Laugier and Sarmah [3] and the references therein). One of these generalizations is the concept of a *near-perfect number*. For a proper divisor d of n, we call n a near-perfect number with *redundant divisor* d if $\sigma(n) = 2n + d$. Pollack and Shevelev [5] and, Ren and Chen [6] found all near-perfect numbers with two distinct prime factors, which we list below.

Type 1. $n = 2^{t-1}(2^t - 2^k - 1)$, where $2^t - 2^k - 1$ is prime, and 2^k is the redundant divisor.

Type 2. $n = 2^p P_p = 2^{2p-1}(2^p - 1)$, where both p and $2^p - 1$ are prime numbers, and $2P_p$ is the redundant divisor.

Type 3. $n = (2^p - 1)P_p = 2^{p-1}(2^p - 1)^2$, where both p and $2^p - 1$ are prime numbers, and $2^p - 1$ is the redundant divisor.

Type 4. This type consists of only the number 40.

Tang, Ren and Li [8] and Tang, Ma and Feng [7] proved that no odd near-perfect numbers exist with three and four distinct odd prime factors respectively. It is an open problem to show that there exist only finitely many odd near-perfect numbers, as well as to classify all the even near-perfect numbers.

Another aspect of modern number theory which has gained a lot of attention is the question of representing integers by sums of powers. In this connection, Waring's problem asks whether each natural number n can be written as a sum of at most s integral powers of k. For instance, it is known that any natural number can be written as the sum of at most 9 integral cubes (see the excellent book by Nathanson [4].) Recently, Farhi [1] showed that for even perfect numbers this bound can be reduced; he proved that any even perfect number can be written as a sum of at most five positive integral cubes and further conjectured that they can be written as a sum of at most three positive integral cubes. This conjecture was later studied by Ulas [9], who although could not prove it but found that even perfect numbers can be written as the sum of three integral cubes. In this paper, we investigate similar questions for even near-perfect numbers with at most two distinct prime factors as well as improve upon a recent result of Ulas [9, Theorem 2.1]. An excellent, but brief review of the existing literature related to this problem can be found in the introduction of Ulas' paper [9].

Our main results are the following.

Theorem 1.1. *If $n \equiv \pm 1 \pmod 6$, then the Diophantine equation*

$$P_n = x^3 + y^3 + z^3, \tag{1.1}$$

has at least three solutions in integers x, y and z.

Theorem 1.2. *If n is an even near-perfect number with two distinct prime factors. Then*

1. *n can be written as a sum of at most seven integral cubes if n is of Type 1, provided $t \not\equiv 0 \pmod 3$.*
2. *n can be written as a sum of at most five integral cubes if n is of Type 2.*
3. *n can be written as a sum of at most six integral cubes if n is of Type 3.*
4. *n can be written as a sum of at most four integral cubes if n is of Type 4.*

The rest of the paper is arranged as follows: Section 2 gives the proofs of Theorems 1.1 and 1.2 and we end this note with some remarks in Section 3.

2 Proof of the Results

2.1 Proof of Theorem 1.1

Before we prove Theorem 1.1 we point out that Ulas [9, Theorem 2.1] has already found two solutions for equation (1.1) when $n \equiv \pm 1 \pmod 6$, which we list below:

$$\begin{aligned} P_{6n+1} &= (2^{n-2}(2^{3n+2} - 21))^3 + (2^{n-2}(2^{3n+2} + 21))^3 - (11.2^{2n-1})^3 \\ &= (2^{4n})^3 + (2^{4n})^3 - (2^{2n})^3, \end{aligned} \tag{2.1}$$

$$\begin{aligned} P_{6n+5} &= (2^n(2^{3(n+1)}+2^{2(n+1)}+1))^3+(2^n(2^{3(n+1)}-2^{2(n+1)}-1))^3-(2^{2(n+1)}(2^{2n+1}+1))^3 \\ &= (2^{2n+1}(2^{2(n+1)}-2^{n+1}-1))^3+(2^{2n+1}(2^{2(n+1)}+2^{n+1}-1))^3-(2^{4n+3})^3. \end{aligned} \tag{2.2}$$

Proof of Theorem 1.1. It is sufficient to give a solution of equation (1.1) for $n \equiv \pm 1 \pmod 6$ which is different from equations (2.1) and (2.2). This is given below:

$$\begin{aligned} P_{6n+1} &= (2^{2n}(2^{2n}+2^n-1))^3+(2^{2n}(2^{2n}-2^n-1))^3+(2^{2n})^3, \\ P_{6n+5} &= (2^{4n+3})^3-(2^{2n+1})^3-(2^{2n+1})^3. \end{aligned}$$

□

Corollary 2.1. *Every even perfect number $n > 28$ can be represented as a sum of three integral cubes in at least three different ways.*

2.2 Proof of Theorem 1.2

If $N = 2^{p-1}(2^p-1)$ is a perfect number then Farhi [1] showed that for $a = n^2+n-1$ and $b = n^2-n-1$, if $p = 6k+1$ for some integer k, then

$$N = (2^{3k}(2^{2k}+2^k-1))^3+(2^{2k}(2^{2k}-2^k-1))^3+(2^{2k})^3, \tag{2.3}$$

and if $p = 6k+5$ for some integer k, then

$$N = (3\cdot 2^{4k+1})^3+(3\cdot 2^{4k+1})^3+(2\cdot 2^{4k+1})^3+(2^{2k+1}(2^{2k}+2^k-1))^3+(2^{2k+1}(2^{2k}-2^k-1))^3. \tag{2.4}$$

To prove the above representations, he used the following elementary identity, which we use freely in the remainder:

$$2q^6-2 = (q^2+q-1)^3+(q^2-q-1)^3. \tag{2.5}$$

We prove Theorem 1.2 in a series of Lemmas below.

Lemma 2.2. *Every even near-perfect number with two distinct prime factors of Type 1 can be written as the sum of at most seven integral cubes, provided $t \not\equiv 0 \pmod 3$.*

Proof. We notice that $t > k$, otherwise n is negative. If $t = 2$, then the only possibility is $k = 1$ for which 2^t-2^k-1 is not a prime. If $t = 3$, then n is either 20 or 12, both of which can be written as the sum of five integral cubes[1]. If $t = 4$, then n is $104, 88$ or 56. Again, all of them can be written as the sum of five integral cubes[2].

Let $t \geq 5$, and we write n as

$$n = 2^{t-1}(2^t-1)-2^{t+k-1}. \tag{2.6}$$

If $t = 6m+1$ for some integer m, then

$$2^{t-1}(2^t-1) = (2^{m-2}(2^{3m+2}-21))^3+(2^{m-2}(2^{3m+2}+21))^3-(11.2^{2m-1})^3. \tag{2.7}$$

From equations (2.6) and (2.7) when $t = 6m+1$, we have

$$n = (2^{m-2}(2^{3m+2}-21))^3+(2^{m-2}(2^{3m+2}+21))^3-(11.2^{2m-1})^3-2^{6m+k},$$

which is the sum of at most four, five or seven integral cubes depending on whether $k \equiv 0, 1$ or $-1 \pmod 3$ respectively.

[1] $20 = 4^3-3^3-2^3-2^3-1^3$ and $12 = 2^3+1^3+1^3+1^3+1^3$.
[2] $104 = 5^3-3^3+2^3-1^3-1^3$, $88 = 4^3+3^3-1^3-1^3-1^3$ and $56 = 7^3-6^3-4^3-2^3+1^3$.

If $t = 6m-1$ for some integer m, then

$$2^{t-1}(2^t-1) = (2^{2m-1}(2^{2m}-2^m-1))^3 + (2^{2m-1}(2^{2m}+2^m-1))^3 - (2^{4m-1})^3. \tag{2.8}$$

From equations (2.6) and (2.8) when $t = 6m-1$, we have

$$n = (2^{2m-1}(2^{2m}-2^m-1))^3 + (2^{2m-1}(2^{2m}+2^m-1))^3 - (2^{4m-1})^3 - 2^{6m+k-2},$$

which is the sum of at most five, seven or four integral cubes depending on whether $k \equiv 0, 1$ or $-1 \pmod 3$ respectively.

If $t = 6m+2$ for some integer m, then

$$2^{t-1}(2^t-1) = (2^{4m+1})^3 - (2^{2m})^3 - (2^{2m})^3. \tag{2.9}$$

From equations (2.6) and (2.9) when $t = 6m+2$, we have

$$n = (2^{4m+1})^3 - (2^{2m})^3 - (2^{2m})^3 - 2^{6m+1+k},$$

which is the sum of at most five, seven or four integral cubes depending on whether $k \equiv 0, 1$ or $-1 \pmod 3$ respectively.

If $t = 6m+4$ for some integer m, then we have

$$n = 2.(2^{4m+2})^3 - 2^{6m+k+3} - (2^{2m+1})^3, \tag{2.10}$$

which is the sum of at most four, five or seven integral cubes depending on whether $k \equiv 0, 1$ or $-1 \pmod 3$ respectively. □

Lemma 2.3. *Even near-perfect numbers with two distinct prime factors of Type 2 can be written as sum of at most five integral cubes.*

Proof. Let $n = 2^{2p-1}(2^p-1)$. If $p = 6m-1$ for some integer m we have

$$n = (2^{4m-2})^3(4\cdot(2^{2m})^3 - 2^3).$$

The above is readily seen to be the sum of five cubes.

If $p = 6m+1$ for some integer m, then we have

$$n = 2^{12m}[2\cdot 2^{6m} + (2.2^{6m}-2)].$$

Now, using equation (2.5) and some simplification we can write the above as

$$n = 2\cdot(2^{6m})^3 + (2^{6m}+2^{5m}-2^{4m})^3 + (2^{6m}-2^{5m}-2^{4m})^3.$$

□

Lemma 2.4. *Even near-perfect numbers with two distinct prime factors of Type 3 can be written as sum of at most six integral cubes.*

Proof. Let $n = 2^{p-1}(2^p-1)^2 = 2^{p-1}(2^{2p}-2^{p+1}+1)$. If $p = 6m+1$ for some integer m, then we have

$$n = 2^{6m}(2^p(2.2^{6m}-2)+1).$$

Using equation (2.5) we get

$$n = 2.2^{12m}((2^{2m}+2^m-1)^3 + (2^{2m}-2^m-1)^3) + 2^{6m},$$

which is a sum of four positive integral cubes.

If $p=6m+5$, then we have

$$n=2^{18m+14}-2^{6m+3}(2.2^{6m+6}-2).$$

Using equation (2.5) we get

$$n=4.2^{18m+12}-2^{6m+3}((2^{2m+2}+2^{m+1}-1)^3+(2^{2m+2}-2^{m+1}-1)^3),$$

which is a sum of six integral cubes. □

Proof of Theorem 1.2. Part (1) follows from Lemma 2.2. Part (2) follows from Lemma 2.3. Part (3) follows from Lemma 2.4. Part (4) is trivial since $40=4^3-2^3-2^3-2^3$. □

3 Remarks

As a side remark, from equation (2.10) we notice that

$$P_{6n+4}=(2^{4m+2})^3+(2^{4m+2})^3-(2^{2m+1})^3.$$

This gives us the following result.

Theorem 3.1. *If $n\equiv 4 \pmod 6$, then the Diophantine equation*

$$P_n=x^3+y^3+z^3, \tag{3.1}$$

has at least one solution in integers x,y and z.

There are several natural questions which we can ask (partly motivated by Farhi's conjecture [1, Conjecture 2.2]).

Question 3.1. It is seen that for Type 3 even near-perfect numbers, when $p=6m+1$ for some integer m then n can be written as a sum of five positive integral cubes. Is this also true when $p=6m+5$ for some integer m?

Question 3.2. What can be said about Type 1 even near-perfect numbers when $t=3m$ for some integer m?

Question 3.3. What can be said about the representation of other classes of generalized perfect numbers as sums of cubes or higher powers?

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